NATIONAL BOARD FOR HIGHER MATHEMATICS A

AND

HOMI BHABHA CENTRE FOR SCIENCE EDUCATION
TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2012

Mumbai Region

October 7, 2012

QUESTION PAPER SET: A

- There are 20 questions in this question paper. Each question carries 5 marks.
- Answer all questions.
- Time allotted: 2 hours.

QUESTIONS

1. Rama was asked by her teacher to subtract 3 from a certain number and then divide the result by 9. Instead, she subtracted 9 and then divided the result by 3. She got 43 as the answer. What would have been her answer if she had solved the problem correctly?

2. A triangle with perimeter 7 has integer side lengths. What is the maximum possible area of such a triangle?

3. For how many pairs of positive integers (x, y) is $x + 3y = 100$?

4. The letters R, M, and O represent whole numbers. If $R \times M \times O = 240$, $R \times O + M = 46$ and $R + M \times O = 64$, what is the value of $R + M + O$?

5. Let $S_n = n^2 + 20n + 12$, n a positive integer. What is the sum of all possible values of n for which S_n is a perfect square?

6. A postman has to deliver five letters to five different houses. Mischievously, he posts one letter through each door without looking to see if it is the correct address. In how many different ways could he do this so that exactly two of the five houses receive the correct letters?

7. In ΔABC, we have $AC = BC = 7$ and $AB = 2$. Suppose that D is a point on line AB such that B lies between A and D and $CD = 8$. What is the length of the segment BD?

8. In rectangle $ABCD$, $AB = 5$ and $BC = 3$. Points F and G are on line segment CD so that $DF = 1$ and $GC = 2$. Lines AF and BG intersect at E. What is the area of ΔAEB?

9. Suppose that $4^{X_1} = 5, 5^{X_2} = 6, 6^{X_3} = 7, \ldots, 126^{X_{123}} = 127, 127^{X_{124}} = 128$. What is the value of the product $X_1 X_2 \ldots X_{124}$?

10. $ABCD$ is a square and $AB = 1$. Equilateral triangles AYB and CXD are drawn such that X and Y are inside the square. What is the length of XY?

11. Let $P(n) = (n+1)(n+3)(n+5)(n+7)(n+9)$. What is the largest integer that is a divisor of $P(n)$ for all positive even integers n?

12. If $\dfrac{1}{\sqrt{2011 + \sqrt{2011^2 - 1}}} = \sqrt{m} - \sqrt{n}$, where m and n are positive integers, what is the value of $m + n$?

13. If $a = b - c$, $b = c - d$, $c = d - a$ and $abcd \neq 0$ then what is the value of $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$?

14. O and I are the circumcentre and incentre of ΔABC respectively. Suppose O lies in the interior of ΔABC and I lies on the circle passing through B, O, and C. What is the magnitude of $\angle BAC$ in degrees?

15. How many non-negative integral values of x satisfy the equation $\left[\frac{x}{5}\right] = \left[\frac{x}{7}\right]$?
(Here $[x]$ denotes the greatest integer less than or equal to x. For example $[3.4] = 3$ and $[-2.3] = -3$.)

16. Let N be the set of natural numbers. Suppose $f : N \to N$ is a function satisfying the following conditions.

    (a) $f(mn) = f(m)f(n)$;

    (b) $f(m) < f(n)$ if $m < n$;

    (c) $f(2) = 2$.

    What is the value of $\sum_{k=1}^{20} f(k)$?

17. Let x_1, x_2, x_3 be the roots of the equation $x^3 + 3x + 5 = 0$. What is the value of the expression

$$\left(x_1 + \frac{1}{x_1}\right)\left(x_2 + \frac{1}{x_2}\right)\left(x_3 + \frac{1}{x_3}\right)?$$

18. What is the sum of the squares of the roots of the equation $x^2 - 7[x] + 5 = 0$?
(Here $[x]$ denotes the greatest integer less than or equal to x. For example $[3.4] = 3$ and $[-2.3] = -3$.)

19. How many integer pairs (x, y) satisfy $x^2 + 4y^2 - 2xy - 2x - 4y - 8 = 0$?

20. PS is a line segment of length 4 and O is the midpoint of PS. A semicircular arc is drawn with PS as diameter. Let X be the midpoint of this arc. Q and R are points on the arc PXS such that QR is parallel to PS and the semicircular arc drawn with QR as diameter is tangent to PS. What is the area of the region $QXROQ$ bounded by the two semicircular arcs?

END OF QUESTION PAPER

A

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TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2013
Mumbai Region

October 20, 2013

QUESTION PAPER SET: A

- There are 20 questions in this question paper. Each question carries 5 marks.
- Answer all questions.
- Time allotted: 2 hours.

QUESTIONS

1. What is the smallest positive integer k such that $k(3^3 + 4^3 + 5^3) = a^n$ for some positive integers a and n, with $n > 1$?

2. Let $S_n = \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}+\sqrt{k}}$. What is the value of $\sum_{n=1}^{99} \frac{1}{S_n + S_{n-1}}$?

3. It is given that the equation $x^2 + ax + 20 = 0$ has integer roots. What is the sum of all possible values of a?

4. Three points X, Y, Z are on a striaght line such that $XY = 10$ and $XZ = 3$. What is the product of all possible values of YZ?

5. There are $n-1$ red balls, n green balls and $n+1$ blue balls in a bag. The number of ways of choosing two balls from the bag that have different colours is 299. What is the value of n?

6. Let $S(M)$ denote the sum of the digits of a positive integer M written in base 10. Let N be the smallest positive integer such that $S(N) = 2013$. What is the value of $S(5N + 2013)$?

7. Let Akbar and Birbal together have n marbles, where $n > 0$.

   Akbar says to Birbal, " If I give you some marbles then you will have twice as many marbles as I will have." Birbal says to Akbar, " If I give you some marbles then you will have thrice as many marbles as I will have."

   What is the minimum possible value of n for which the above statements are true?

8. Let AD and BC be the parallel sides of a trapezium $ABCD$. Let P and Q be the midpoints of the diagonals AC and BD. If $AD = 16$ and $BC = 20$, what is the length of PQ?

A

9. In a triangle ABC, let H, I and O be the orthocentre, incentre and circumcentre, respectively. If the points B, H, I, C lie on a circle, what is the magnitude of $\angle BOC$ in degrees?

10. Carol was given three numbers and was asked to add the largest of the three to the product of the other two. Instead, she multiplied the largest with the sum of the other two, but still got the right answer. What is the sum of the three numbers?

11. Three real numbers x, y, z are such that $x^2 + 6y = -17$, $y^2 + 4z = 1$ and $z^2 + 2x = 2$. What is the value of $x^2 + y^2 + z^2$?

12. Let ABC be an equilateral triangle. Let P and S be points on AB and AC, respectively, and let Q and R be points on BC such that $PQRS$ is a rectangle. If $PQ = \sqrt{3}PS$ and the area of $PQRS$ is $28\sqrt{3}$, what is the length of PC?

13. To each element of the set $S = \{1, 2, \ldots, 1000\}$ a colour is assigned. Suppose that for any two elements a, b of S, if 15 divides $a + b$ then they are both assigned the same colour. What is the maximum possible number of distinct colours used?

14. Let m be the smallest odd positive integer for which $1 + 2 + \cdots + m$ is a square of an integer and let n be the smallest even positive integer for which $1 + 2 + \cdots + n$ is a square of an integer. What is the value of $m + n$?

15. Let A_1, B_1, C_1, D_1 be the midpoints of the sides of a convex quadrilateral $ABCD$ and let A_2, B_2, C_2, D_2 be the midpoints of the sides of the quadrilateral $A_1B_1C_1D_1$. If $A_2B_2C_2D_2$ is a rectangle with sides 4 and 6, then what is the product of the lengths of the diagonals of $ABCD$?

16. Let $f(x) = x^3 - 3x + b$ and $g(x) = x^2 + bx - 3$, where b is a real number. What is the sum of all possible values of b for which the equations $f(x) = 0$ and $g(x) = 0$ have a common root?

17. Let S be a circle with centre O. A chord AB, not a diameter, divides S into two regions R_1 and R_2 such that O belongs to R_2. Let S_1 be a circle with centre in R_1, touching AB at X and S internally. Let S_2 be a circle with centre in R_2, touching AB at Y, the circle S internally and passing through the centre of S. The point X lies on the diameter passing through the centre of S_2 and $\angle YXO = 30°$. If the radius of S_2 is 100 then what is the radius of S_1?

18. What is the maximum possible value of k for which 2013 can be written as a sum of k consecutive positive integers?

19. In a triangle ABC with $\angle BCA = 90°$, the perpendicular bisector of AB intersects segments AB and AC at X and Y, respectively. If the ratio of the area of quadrilateral $BXYC$ to the area of triangle ABC is $13 : 18$ and $BC = 12$ then what is the length of AC?

20. What is the sum (in base 10) of all the natural numbers less than 64 which have exactly three ones in their base 2 representation?

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TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2014
Mumbai Region

October 12, 2014

QUESTION PAPER SET: A

- There are 20 questions in this question paper. Each question carries 5 marks.
- Answer all questions.
- Time allotted: 2.5 hours.

QUESTIONS

1. A natural number k is such that $k^2 < 2014 < (k+1)^2$. What is the largest prime factor of k?

2. The first term of a sequence is 2014. Each succeeding term is the sum of the cubes of the digits of the previous term. What is the 2014^{th} term of the sequence?

3. Let $ABCD$ be a convex quadrilateral with perpendicular diagonals. If $AB = 20, BC = 70$ and $CD = 90$, then what is the value of DA?

4. In a triangle with integer side lengths, one side is three times as long as a second side, and the length of the third side is 17. What is the greatest possible perimeter of the triangle?

5. If real numbers a, b, c, d, e satisfy

$$a+1 = b+2 = c+3 = d+4 = e+5 = a+b+c+d+e+3,$$

what is the value of $a^2+b^2+c^2+d^2+e^2$?

6. What is the smallest possible natural number n for which the equation $x^2 - nx + 2014 = 0$ has integer roots?

7. If $x^{(x^4)} = 4$, what is the value of $x^{(x^2)} + x^{(x^8)}$?

8. Let S be a set of real numbers with mean M. If the means of the sets $S \cup \{15\}$ and $S \cup \{15, 1\}$ are $M+2$ and $M+1$, respectively, then how many elements does S have?

9. Natural numbers k, l, p and q are such that if a and b are roots of $x^2 - kx + l = 0$ then $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are the roots of $x^2 - px + q = 0$. What is the sum of all possible values of q?

10. In a triangle ABC, X and Y are points on the segments AB and AC, respectively, such that $AX : XB = 1 : 2$ and $AY : YC = 2 : 1$. If the area of triangle AXY is 10 then what is the area of triangle ABC?

11. For natural numbers x and y, let (x, y) denote the greatest common divisor of x and y. How many pairs of natural numbers x and y with $x \leq y$ satisfy the equation $xy = x + y + (x, y)$?

12. Let $ABCD$ be a convex quadrilateral with $\angle DAB = \angle BDC = 90^\circ$. Let the incircles of triangles ABD and BCD touch BD at P and Q, respectively, with P lying in between B and Q. If $AD = 999$ and $PQ = 200$ then what is the sum of the radii of the incircles of triangles ABD and BDC?

13. For how many natural numbers n between 1 and 2014 (both inclusive) is $\dfrac{8n}{9999-n}$ an integer?

14. One morning, each member of Manjul's family drank an 8-ounce mixture of coffee and milk. The amounts of coffee and milk varied from cup to cup, but were never zero. Manjul drank 1/7-th of the total amount of milk and 2/17-th of the total amount of coffee. How many people are there in Manjul's family?

15. Let XOY be a triangle with $\angle XOY = 90^\circ$. Let M and N be the midpoints of legs OX and OY, respectively. Suppose that $XN = 19$ and $YM = 22$. What is XY?

16. In a triangle ABC, let I denote the incenter. Let the lines AI, BI and CI intersect the incircle at P, Q and R, respectively. If $\angle BAC = 40^\circ$, what is the value of $\angle QPR$ in degrees?

17. For a natural number b, let $N(b)$ denote the number of natural numbers a for which the equation $x^2 + ax + b = 0$ has integer roots. What is the smallest value of b for which $N(b) = 20$?

18. Let f be a one-to-one function from the set of natural numbers to itself such that $f(mn) = f(m)f(n)$ for all natural numbers m and n. What is the least possible value of $f(999)$?

19. Let $x_1, x_2, \cdots, x_{2014}$ be real numbers different from 1, such that $x_1 + x_2 + \cdots + x_{2014} = 1$ and
$$\frac{x_1}{1-x_1} + \frac{x_2}{1-x_2} + \cdots + \frac{x_{2014}}{1-x_{2014}} = 1\,.$$
What is the value of
$$\frac{x_1^2}{1-x_1} + \frac{x_2^2}{1-x_2} + \frac{x_3^2}{1-x_3} + \cdots + \frac{x_{2014}^2}{1-x_{2014}}\,?$$

20. What is the number of ordered pairs (A, B) where A and B are subsets of $\{1, 2, \ldots, 5\}$ such that neither $A \subseteq B$ nor $B \subseteq A$?

Pre-RMO 2014 results will be put up on HBCSE website by 31/10/2014.

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TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2015
Mumbai Region

October 4, 2015

QUESTION PAPER SET: A

- There are 20 questions in this question paper. Each question carries 5 marks.
- Answer all questions.
- Time allotted: 2.5 hours.

QUESTIONS

1. A man walks a certain distance and rides back in $3\frac{3}{4}$ hours; he could ride both ways in $2\frac{1}{2}$ hours. How many hours would it take him to walk both ways? [**5**]

2. Positive integers a and b are such that $a + b = a/b + b/a$. What is the value of $a^2 + b^2$? [**2**]

3. The equations $x^2 - 4x + k = 0$ and $x^2 + kx - 4 = 0$, where k is a real number, have exactly one common root. What is the value of k? [**3**]

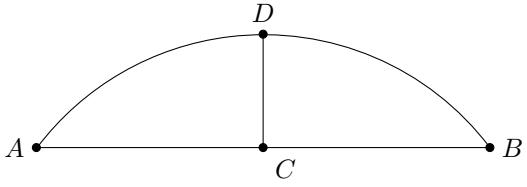
4. Let $P(x)$ be a non-zero polynomial with integer coefficients. If $P(n)$ is divisible by n for each positive integer n, what is the value of $P(0)$? [**0**]

5. How many line segments have both their endpoints located at the vertices of a given cube? [**28**]

6. Let $E(n)$ denote the sum of the even digits of n. For example, $E(1243) = 2 + 4 = 6$. What is the value of $E(1) + E(2) + E(3) + \cdots + E(100)$? [**400**]

7. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square? [**8**]

8. The figure below shows a broken piece of a circular plate made of glass.



C is the midpoint of AB, and D is the midpoint of arc AB. Given that $AB = 24$ cm and $CD = 6$ cm, what is the radius of the plate in centimetres? (The figure is not drawn to scale.) [**15**]

9. A 2×3 rectangle and a 3×4 rectangle are contained within a square without overlapping at any interior point, and the sides of the square are parallel to the sides of the two given rectangles. What is the smallest possible area of the square? [**25**]

10. What is the greatest possible perimeter of a right-angled triangle with integer side lengths if one of the sides has length 12 ? [**84**]

11. In rectangle $ABCD$, $AB = 8$ and $BC = 20$. Let P be a point on AD such that $\angle BPC = 90°$. If r_1, r_2, r_3 are the radii of the incircles of triangles APB, BPC and CPD, what is the value of $r_1 + r_2 + r_3$? [**8**]

12. Let a, b, and c be real numbers such that $a - 7b + 8c = 4$ and $8a + 4b - c = 7$. What is the value of $a^2 - b^2 + c^2$? [**1**]

13. Let n be the largest integer that is the product of exactly 3 distinct prime numbers, x, y and $10x + y$, where x and y are digits. What is the sum of the digits of n? [**12**]

14. At a party, each man danced with exactly four women and each woman danced with exactly three men. Nine men attended the party. How many women attended the party? [**12**]

15. If $3^x + 2^y = 985$ and $3^x - 2^y = 473$, what is the value of xy? [**48**]

16. In acute-angled triangle ABC, let D be the foot of the altitude from A, and E be the midpoint of BC. Let F be the midpoint of AC. Suppose $\angle BAE = 40°$. If $\angle DAE = \angle DFE$, what is the magnitude of $\angle ADF$ in degrees? [**40**]

17. A subset B of the set of first 100 positive integers has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B? [**62**]

18. Let a, b and c be such that $a + b + c = 0$ and

$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}$$

is defined. What is the value of P? [**1**]

19. The circle ω touches the circle Ω internally at P. The centre O of Ω is outside ω. Let XY be a diameter of Ω which is also tangent to ω. Assume $PY > PX$. Let PY intersect ω at Z. If $YZ = 2PZ$, what is the magnitude of $\angle PYX$ in degrees? [**15**]

20. The digits of a positive integer n are four consecutive integers in decreasing order when read from left to right. What is the sum of the possible remainders when n is divided by 37? [**217**]

NATIONAL BOARD FOR HIGHER MATHEMATICS
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TATA INSTITUTE OF FUNDAMENTAL RESEARCH

Pre-REGIONAL MATHEMATICAL OLYMPIAD, 2015
Mumbai Region

October 4, 2015

QUESTION PAPER SET: B

- There are 20 questions in this question paper. Each question carries 5 marks.
- Answer all questions.
- Time allotted: 2.5 hours.

QUESTIONS

1. A man walks a certain distance and rides back in $3\frac{3}{4}$ hours; he could ride both ways in $2\frac{1}{2}$ hours. How many hours would it take him to walk both ways? [**5**]

2. The equations $x^2 - 4x + k = 0$ and $x^2 + kx - 4 = 0$, where k is a real number, have exactly one common root. What is the value of k? [**3**]

3. Positive integers a and b are such that $a + b = a/b + b/a$. What is the value of $a^2 + b^2$? [**2**]

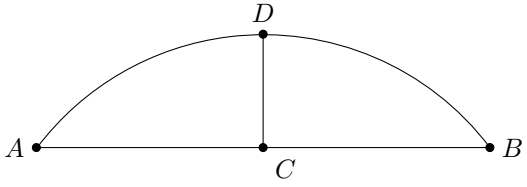
4. How many line segments have both their endpoints located at the vertices of a given cube? [**28**]

5. Let $P(x)$ be a non-zero polynomial with integer coefficients. If $P(n)$ is divisible by n for each positive integer n, what is the value of $P(0)$? [**0**]

6. How many two-digit positive integers N have the property that the sum of N and the number obtained by reversing the order of the digits of N is a perfect square? [**8**]

7. Let $E(n)$ denote the sum of the even digits of n. For example, $E(1243) = 2 + 4 = 6$. What is the value of $E(1) + E(2) + E(3) + \cdots + E(100)$? [**400**]

8. The figure below shows a broken piece of a circular plate made of glass.



C is the midpoint of AB, and D is the midpoint of arc AB. Given that $AB = 24$ cm and $CD = 6$ cm, what is the radius of the plate in centimetres? (The figure is not drawn to scale.) [**15**]

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$$P = \frac{a^2}{2a^2 + bc} + \frac{b^2}{2b^2 + ca} + \frac{c^2}{2c^2 + ab}$$

is defined. What is the value of P? [**1**]

18. A subset B of the set of first 100 positive integers has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B? [**62**]

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National Board for Higher Mathematics

Homi Bhabha Center for Science Education, TIFR

Pre Regional Mathematical Olympiad, 2016

Delhi Region

Set A

Maximum Marks: 50

Important Note: The answer to each question is an integer between 0 and 2016. Each Candidate must write the final answer (in the space provided) as, Final answer = Correct answer + sum of all the digits of their roll number. Only the final answer shall be considered.

Problem 1. Consider the sequence $1, 3, 3, 3, 5, 5, 5, 5, 5, 7, 7, 7, 7, 7, 7, 7, ...$ and evaluate its 2016th term. [3 marks]

Problem 2. The five digit number $2a9b1$ is a perfect square. Find the value of $a^{b-1} + b^{a-1}$. [3 marks]

Problem 3. The date index of a date is defined as ($12 \times$ month number + day number). Three events each with a frequency of once in 21 days, 32 days and 9 days, respectively, occurred simultaneously for the first time on July 31, 1961 (Ireland joining the European Economic Community). Find the date index of the date when they occur simultaneously for the eleventh time. [3 marks]

Problem 4. There are three kinds of fruits in the market. How many ways are there to purchase 25 fruits from among them if each kind has at least 25 of its fruit available? [3 marks]

Problem 5. In a school there are 500 students. Two-thirds of the students who do not wear glasses, do not bring lunch. Three-quarters of the students who do not bring lunch do not wear glasses. Altogether, 60 students who wear glasses bring lunch. How many students do not wear glasses and do not bring lunch? [3 marks]

Problem 6. Let AD be an altitude in a right triangle ABC with $\angle A = 90^\circ$ and D on BC. Suppose that the radii of the incircles of the triangles ABD and ACD are 33 and 56 respectively. Let r be the radius of the incircle of triangle ABC. Find the value of $3(r + 7)$. [3 marks]

Or

Problem 6'. Find the sum of digits in decimal form of the number $(999\ldots9)^3$. (There are 12 nines) [3 marks]

Problem 7. Let $s(n)$ and $p(n)$ denote the sum of all digits of n and the product of all digits of n (when written in decimal form), respectively. Find the sum of all two-digit natural numbers n such that $n = s(n) + p(n)$. [3 marks]

Problem 8. Suppose that a and b are real numbers such that $ab \neq 1$ and the equations $120a^2 - 120a + 1 = 0$ and $b^2 - 120b + 120 = 0$ hold. Find the value of $\frac{1+b+ab}{a}$. [3 marks]

Problem 9. Between 5pm and 6pm, I looked at my watch. Mistaking the hour hand for the minute hand and the minute hand for the hour hand, I mistook the time to be 57 minutes earlier than the actual time. Find the number of minutes past 5 when I looked at my watch. [3 marks]

Problem 10. In triangle ABC right angled at vertex B, a point O is chosen on the side BC such that the circle γ centered at O of radius OB touches the side AC. Let $AB = 63$ and $BC = 16$, and the radius of γ be of the form $\frac{m}{n}$ where m, n are relatively prime positive integers. Find the value of $m + n$. [3 marks]

Problem 11. Consider the 50 term sums:

$$S = \frac{1}{1 \times 2} + \frac{1}{3 \times 4} + \cdots + \frac{1}{99 \times 100}$$

$$T = \frac{1}{51 \times 100} + \frac{1}{52 \times 99} + \cdots + \frac{1}{100 \times 51}$$

The ratio $\frac{S}{T}$ is written in the lowest form $\frac{m}{n}$ where m, n are relatively prime natural numbers. Find the value of $m + n$. [4 marks]

Problem 12. Find the value of the expression

$$\frac{(3^4 + 3^2 + 1) \cdot (5^4 + 5^2 + 1) \cdot (7^4 + 7^2 + 1) \cdot (9^4 + 9^2 + 1) \cdot (11^4 + 11^2 + 1) \cdot (13^4 + 13^2 + 1)}{(2^4 + 2^2 + 1) \cdot (4^4 + 4^2 + 1) \cdot (6^4 + 6^2 + 1) \cdot (8^4 + 8^2 + 1) \cdot (10^4 + 10^2 + 1) \cdot (12^4 + 12^2 + 1)}$$

when written in lowest form. [4 marks]

Problem 13. The hexagon $OLYMPI$ has a reflex angle at O and convex at every other vertex. Suppose that $LP = 3\sqrt{2}$ units and the condition

$$\angle O = 10\angle L = 2\angle Y = 5\angle M = 2\angle P = 10\angle I$$

holds. Find the area (in sq units) of the hexagon. [4 marks]

Problem 14. A natural number a has four digits and a^2 ends with the same four digits as that of a. Find the value of $(10,080 - a)$. [4 marks]

Problem 15. Points G and O denote the centroid and the circumcenter of the triangle ABC. Suppose that $\angle AGO = 90^\circ$ and $AB = 17$,$AC = 19$. Find the value of BC^2. [4 marks]

PRMO 2017

1. How many positive integers less than 1000 have the property that the sum of the digits of each such number is divisible by 7 and the number itself is divisible by 3?

2. Suppose a, b are positive real numbers such that $a\sqrt{a} + b\sqrt{b} = 183, \quad a\sqrt{b} + b\sqrt{a} = 182.$ Find $\frac{9}{5}(a + b)$.

3. A contractor has two teams of workers: team A and team B. Team A can complete a job in 12 days and team B can do the same job in 36 days. Team A starts working on the job and team B joins team A after four days. The team A withdraws after two more days. For how many more days should team B work to complete the job?

4. Let a, b be integers such that all the roots of the equation $(x^2 + ax + 20)(x^2 + 17x + b) = 0$ are negative integers. What is the smallest possible value of $a + b$?

5. Let u, v, w be real numbers in geometric progression such that $u > v > w$. Suppose $u^{40} = v^n = w^{60}$. Find the value of n.

6. Let the sum $\sum_{n=1}^{9} \frac{1}{n(n+1)(n+2)}$ written in its lowest terms be $\frac{p}{q}$. Find the value of $q - p$.

7. Find the number of positive integers n, such that $\sqrt{n} + \sqrt{n+1} < 11$.

8 A pen costs ₹11 and a notebook costs ₹13. Find the number of ways in which a person can spend exactly ₹1000 to buy pens and notebooks.

9. There are five cities A, B, C, D, E on a certain island. Each city is connected to every other city by road. In how many ways can a person starting from city A come back to A after visiting some cities without visiting a city more than once and without taking the same road more than once? (The order in which he visits the cities also matters: e.g., the routes $A \to B \to C \to A$ and $A \to C \to B \to A$ are different.)

10. There are eight rooms on the first floor of a hotel, with four rooms on each side of the corridor, symmetrically situated (that is each room is exactly opposite to one other room). Four guests have to be accommodated in four of the eight rooms (that is, one in each) such that no two guests are in adjacent rooms or in opposite rooms. In how many ways can the guests be accommodated?

11. Let $f(x) = \sin\frac{x}{3} + \cos\frac{3x}{10}$ for all real x. Find the least natural number n such that $f\big(n\pi + x\big) = f(x)$ for all real x.

12. In a class, the total numbers of boys and girls are in the ratio $4 : 3$. On one day it was found that 8 boys and 14 girls were absent from the class, and that the number of boys was the square of the number of girls. What is the total number of students in the class?

13. In a rectangle $ABCD$, E is the midpoint of AB; F is a point on AC such that BF is perpendicular to AC; and FE perpendicular to BD. Suppose $BC = 8\sqrt{3}$. Find AB.

14. Suppose x is a positive real number such that $\{x\}, [x]$ and x are in a geometric progression. Find the least positive integer n such that $x^n > 100$. (Here $[x]$ denotes the integer part of x and $\{x\} = x - [x]$.)

15. Integers $1, 2, 3, \ldots, n$, where $n > 2$, are written on a board. Two numbers m, k such

that $1 < m < n$, $1 < k < n$ are removed and the average of the remaining numbers is found to be 17. What is the maximum sum of the two removed numbers?

16. Five distinct 2-digit numbers are in a geometric progression. Find the middle term.

17. Suppose the altitudes of a triangle are 10, 12 and 15. What is its semi-perimeter?

18. If the real numbers x, y, z are such that $x^2 + 4y^2 + 16z^2 = 48$ and $xy + 4yz + 2zx = 24$, what is the value of $x^2 + y^2 + z^2$?

19. Suppose $1, 2, 3$ are the roots of the equation $x^4 + ax^2 + bx = c$. Find the value of c.

20. What is the number of triples (a, b, c) of positive integers such that (i) $a < b < c < 10$ and (ii) $a, b, c, 10$ form the sides of a quadrilateral?

21. Find the number of ordered triples (a, b, c) of positive integers such that $abc = 108$.

22. Suppose in the plane 10 pairwise nonparallel lines intersect one another. What is the maximum possible number of polygons (with finite areas) that can be formed?

23. Suppose an integer x, a natural number n and a prime number p satisfy the equation $7x^2 - 44x + 12 = p^n$. Find the largest value of p.

24. Let P be an interior point of a triangle ABC whose sidelengths are 26, 65, 78. The line through P parallel to BC meets AB in K and AC in L. The line through P parallel to CA meets BC in M and BA in N. The line through P parallel to AB meets CA in S and CB in T. If KL, MN, ST are of equal lengths, find this common length.

25. Let $ABCD$ be a rectangle and let E and F be points on CD and BC respectively such that area$(ADE) = 16$, area$(CEF) = 9$ and area$(ABF) = 25$. What is the area of triangle AEF?

26. Let AB and CD be two parallel chords in a circle with radius 5 such that the centre O lies between these chords. Suppose $AB = 6$, $CD = 8$. Suppose further that the area of the part of the circle lying between the chords AB and CD is $(m\pi + n)/k$, where m, n, k are positive integers with $\gcd(m, n, k) = 1$. What is the value of $m + n + k$?

27. Let Ω_1 be a circle with centre O and let AB be a diameter of Ω_1. Let P be a point on the segment OB different from O. Suppose another circle Ω_2 with centre P lies in the interior of Ω_1. Tangents are drawn from A and B to the circle Ω_2 intersecting Ω_1 again at A_1 and B_1 respectively such that A_1 and B_1 are on the opposite sides of AB. Given that $A_1B = 5$, $AB_1 = 15$ and $OP = 10$, find the radius of Ω_1.

28. Let p, q be prime numbers such that $n^{3pq} - n$ is a multiple of $3pq$ for **all** positive integers n. Find the least possible value of $p + q$.

29. For each positive integer n, consider the highest common factor h_n of the two numbers $n! + 1$ and $(n + 1)!$. For $n < 100$, find the largest value of h_n.

30. Consider the areas of the four triangles obtained by drawing the diagonals AC and BD of a trapezium $ABCD$. The product of these areas, taken two at time, are computed. If among the six products so obtained, two products are 1296 and 576, determine the square root of the maximum possible area of the trapezium to the nearest integer.

MTA-IAPT Pre-Regional Mathematical Olympiad(PRMO),2018

Date: August 19, 2018 **Time: 10 AM to 1 PM**

Number of Questions:30 Max Marks: 102

INSTRUCTIONS

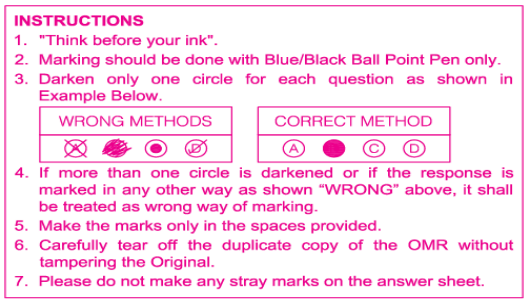
1. Use of mobile phones, smartphones, ipads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a black pencil or a black or blue ball pen. Darken the bubbles completely only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your PRMO score.

4. Incomplete/Incorrectly and carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.


INSTRUCTIONS
1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.
WRONG METHODS
CORRECT METHOD
A B C D
4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.

6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be cosidered as your final answer.

7. Questions 1 to 6 carry 2 marks each; questions 7 to 21 carry 3 marks each; questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

(हिंदी दिशानिर्देश पुस्तिका के अंतिम पृष्ठ पर हैं)

1. A book is published in three volumes, the pages being numbered from 1 onwards. The page numbers are continued from the first volume to the second volume to the third. The number of pages in the second volume is 50 more than that in the first volume, and the number pages in the third volume is one and a half times that in the second. The sum of the page numbers on the first pages of the three volumes is 1709. If n is the last page number, what is the largest prime factor of n?

2. In a quadrilateral $ABCD$, it is given that $AB = AD = 13, BC = CD = 20, BD = 24$. If r is the radius of the circle inscribable in the quadrilateral, then what is the integer closest to r?

3. Consider all 6-digit numbers of the form $abccba$ where b is odd. Determine the number of all such 6-digit numbers that are divisible by 7.

4, The equation $166 \times 56 = 8590$ is valid in some base $b \geq 10$ (that is, $1, 6, 5, 8, 9, 0$ are digits in base b in the above equation). Find the sum of all possible values of $b \geq 10$ satisfying the equation.

5. Let $ABCD$ be a trapezium in which $AB \parallel CD$ and $AD \perp AB$. Suppose $ABCD$ has an incircle which touches AB at Q and CD at P. Given that $PC = 36$ and $QB = 49$, find PQ.

6. Integers a, b, c satisfy $a + b - c = 1$ and $a^2 + b^2 - c^2 = -1$. What is the sum of all possible values of $a^2 + b^2 + c^2$?

SPACE FOR ROUGH WORK

1. एक पुस्तक तीन खंडों में प्रकाशित है, और कुल पृष्ठ संख्या 1 से शुरू होती है। दूसरे खंड की पृष्ठ संख्या पहले खंड के आगे से शुरू होती है व तीसरे की दूसरे के आगे से। दूसरे खंड में पहले खंड से 50 अधिक पृष्ठ हैं, व तीसरे खंड में दूसरे से डेढ़ गुना पृष्ठ हैं। तीनों खंडों के पृथम पृष्ठ की पृष्ठ संख्या का योग 1709 है। अगर n अंतिम पृष्ठ संख्या है, तो n को विभाजित करने वाली सबसे बड़ी अभाज्य संख्या कौनसी है?

2. एक चतुर्भुज $ABCD$ में ये दिया हुआ है कि $AB = AD = 13, BC = CD = 20$ व $BD = 24$ है। अगर r इस चतुर्भुज के अंदर बनाए जा सकने वाले अन्तःवृत्त की त्रिज्या है, तो r से निकटम पूर्णांक का मान क्या होगा?

3. ऐसी 6-अंकों की संख्या $abccba$ के बारे में सोचो जिनमें b विषम है। ऐसी कितनी 6-अंकों की संख्याएँ होंगीं जो कि 7 से विभाजित हो जाती हैं?

4. समीकरण $166 \times 56 = 8590$ किसी आधार (base) $b \geq 10$ में सही है (मतलब कि $1, 6, 5, 8, 9, 0$ आधार (base) b में अंक हैं)। ऐसी सभी सम्भव संख्याओं $b \geq 10$ का योग क्या होगा?

5. $ABCD$ एक समलंब चतुर्भुज है जिसमें कि $AB \parallel CD$ व $AD \perp AB$। मान लो कि इस चतुर्भुज का एक अंतःवृत्त है। मान लो कि यह अंतःवृत्त AB से Q में व CD से P में मिलता है। अगर $PC = 36$ व $QB = 49$ तो PQ का मान क्या होगा?

6. a, b, c ऐसे पूर्णांक हैं जिनके लिए $a + b - c = 1$ व $a^2 + b^2 - c^2 = -1$. $a^2 + b^2 + c^2$ के जो भी मान संभव हैं, उनका योग क्या होगा?

रफ़ कार्य के लिए जगह

7. A point P in the interior of a regular hexagon is at distances 8,8,16 units from three consecutive vertices of the hexagon, respectively. If r is radius of the circumscribed circle of the hexagon, what is the integer closest to r?

8. Let AB be a chord of a circle with centre O. Let C be a point on the circle such that $\angle ABC = 30^\circ$ and O lies inside the triangle ABC. Let D be a point on AB such that $\angle DCO = \angle OCB = 20^\circ$. Find the measure of $\angle CDO$ in degrees.

9. Suppose a, b are integers and $a+b$ is a root of $x^2+ax+b=0$. What is the maximum possible value of b^2?

10. In a triangle ABC, the median from B to CA is perpendicular to the median from C to AB. If the median from A to BC is 30, determine $(BC^2+CA^2+AB^2)/100$.

11. There are several tea cups in the kitchen, some with handles and the others without handles. The number of ways of selecting two cups without a handle and three with a handle is exactly 1200. What is the maximum possible number of cups in the kitchen?

SPACE FOR ROUGH WORK

7. बिंदु P एक सम-षट्भुज का भीतरी बिंदु है और षट्भुज के तीन क्रमानुगत कोनों से उसकी दूरी क्रमशः $8, 8$ व 16 है। अगर r षट्भुज के परिवृत्त की त्रिज्या का मान है तो r के सबसे क़रीब कौनसा पूर्णांक होगा?

8. AB एक वृत्त की जीवा है जिसका केंद्र O है। मान लो कि C वृत्त पर एक ऐसा बिंदु है जिससे कि $\angle ABC = 30^\circ$ व O त्रिभुज ABC के अंदर है। मान लो कि D वृत्त AB पर एक ऐसा बिंदु है जिससे कि $\angle DCO = \angle OCB = 20^\circ$। $\angle CDO$ का मान डिग्री में पता करो।

9. मान लो कि $a + b$ समीकरण $x^2 + ax + b = 0$ का एक हल है। b^2 का अधिकतम सम्भव मान क्या है?

10. एक त्रिभुज ABC में B से CA तक की मध्यिका C से AB तक की मध्यिका से लम्ब है। अगर A से BC तक की मध्यिका की लम्बाई 30 है तो $(BC^2 + CA^2 + AB^2)/100$ का मान ज्ञात करो।

11. किचन में कई चाय के कप हैं, कुछ में हैंडल है, और कुछ में नहीं है। अगर इनमें से दो कप बिना हैंडल के व तीन कप हैंडल के चुनने हों तो यह 1200 तरीकों से किया जा सकता है। किचन में कितने कप हैं?

रफ़ कार्य के लिए जगह

12. Determine the number of 8-tuples $(\epsilon_1, \epsilon_2, \cdots, \epsilon_8)$ such that $\epsilon_1, \epsilon_2, \cdots \epsilon_8 \in \{1, -1\}$ and

$$\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + \cdots + 8\epsilon_8$$

is a multiple of 3.

13. In a triangle ABC, right-angled at A, the altitude through A and the internal bisector of $\angle A$ have lengths 3 and 4, respectively. Find the length of the median through A.

14. If $x = \cos 1^\circ \cos 2^\circ \cos 3^\circ \cdots \cos 89^\circ$ and $y = \cos 2^\circ \cos 6^\circ \cos 10^\circ \cdots \cos 86^\circ$, then what is the integer nearest to $\frac{2}{7} \log_2(y/x)$?

15. Let a and b be natural numbers such that $2a - b, a - 2b$ and $a + b$ are all distinct squares. What is the smallest possible value of b?

16. What is the value of

$$\sum_{\substack{1 \le i < j \le 10 \\ i+j = \text{odd}}} (i + j) - \sum_{\substack{1 \le i < j \le 10 \\ i+j = \text{even}}} (i + j) \, ?$$

SPACE FOR ROUGH WORK

12. ऐसे कितने क्रमवार-समूह्य $(\epsilon_1, \epsilon_2, \cdots, \epsilon_8)$ हैं जिसमें $\epsilon_1, \epsilon_2, \cdots \epsilon_8 \in \{-1, +1\}$ व $\epsilon_1 + 2\epsilon_2 + 3\epsilon_3 + \cdots + 8\epsilon_8$ का मान 3 से भाज्य हो?

13. एक त्रिभुज ABC, जिसमें कोण A समकोण है, ऐसा है कि A से लम्ब की लम्बाई व $\angle A$ के कोण-समद्विभाजक की लम्बाई क्रमशः 3 व 4 है। A से मध्यिका की लम्बाई कितनी होगी?

14. अगर $x = \cos 1^\circ \cos 2^\circ \cos 3^\circ \cdots \cos 89^\circ$ व $y = \cos 2^\circ \cos 6^\circ \cos 10^\circ \cdots \cos 86^\circ$ तो संख्या $\frac{2}{7} \log_2(y/x)$ के सबसे क़रीब कौनसा पूर्णांक होगा?

15. मान लो कि a व b ऐसी प्राकृतिक संख्या हैं जिससे कि $2a - b, a - 2b$ व $a + b$ सभी अलग-अलग पूर्णांकों के वर्ग हैं। b का न्यूनतम सम्भव मान क्या होगा?

16. निम्न का मान पता करो:
$$\sum_{\substack{1 \le i < j \le 10 \\ i+j=\text{odd}}} (i+j) - \sum_{\substack{1 \le i < j \le 10 \\ i+j=\text{even}}} (i+j)?$$

रफ़ कार्य के लिए जगह

17. Triangles ABC and DEF are such that $\angle A = \angle D$, $AB = DE = 17$, $BC = EF = 10$ and $AC - DF = 12$. What is $AC + DF$?

18. If $a, b, c \geq 4$ are integers, not all equal, and $4abc = (a+3)(b+3)(c+3)$, then what is the value of $a + b + c$?

19. Let $N = 6 + 66 + 666 + \cdots + 666\cdots66$, where there are hundred 6's in the last term in the sum. How many times does the digit 7 occur in the number N?

20. Determine the sum of all possible positive integers n, the product of whose digits equals $n^2 - 15n - 27$.

21. Let ABC be an acute-angled triangle and let H be its orthocentre. Let G_1, G_2 and G_3 be the centroids of the triangles HBC, HCA and HAB, respectively. If the area of triangle $G_1G_2G_3$ is 7 units, what is the area of triangle ABC?

SPACE FOR ROUGH WORK

17. ABC व DEF ऐसे त्रिभुज हैं कि $\angle A = \angle D, AB = DE = 17, BC = EF = 10$ और $AC - DF = 12$ है। $AC + DF$ का मान क्या है?

18. अगर $a, b, c \geq 4$ पूर्णांक हैं, सभी बराबर नहीं हैं, और $4abc = (a+3)(b+3)(c+3)$ तो $a+b+c$ का मान क्या है?

19. मान लो कि $N = 6 + 66 + 666 + \cdots + 666\cdots66$ जहाँ आख़िरी संख्या में सौ 6 के अंक हैं। N में अंक 7 कितनी बार आएगा?

20. ऐसे सभी धनात्मक पूर्णांकों का योग पता करो जिनके अंकों का गुणनफल $n^2 - 15n - 27$ है।

21. मान लो कि ABC एक न्यूनकोण त्रिभुज है और H उसका लंबकेंद्र है। मान लो कि G_1, G_2 व G_3 क्रमशः त्रिभुज HBC, HCA व HAB के केंद्रक हैं। अगर त्रिभुज $G_1G_2G_3$ का क्षेत्रफल 7 है, तो त्रिभुज ABC का क्षेत्रफल कितना होगा?

रफ़ कार्य के लिए जगह

22. A positive integer k is said to be *good* if there exists a partition of $\{1, 2, 3, \ldots, 20\}$ in to disjoint proper subsets such that the sum of the numbers in each subset of the partition is k. How many *good* numbers are there?

23. What is the largest positive integer n such that

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n(a+b+c)$$

holds for all positive real numbers a, b, c.

24. If N is the number of triangles of different shapes (i.e., not similar) whose angles are all integers (in degrees), what is $N/100$?

25. Let T be the smallest positive integer which, when divided by $11, 13, 15$ leaves remainders in the sets $\{7, 8, 9\}$, $\{1, 2, 3\}$, $\{4, 5, 6\}$ respectively. What is the sum of the squares of the digits of T?

26. What is the number of ways in which one can choose 60 unit squares from a 11×11 chessboard such that no two chosen squares have a side in common?

SPACE FOR ROUGH WORK

22. एक पूर्णांक k को हम अच्छा कहेंगे अगर $\{1, 2, 3, \ldots, 20\}$ को हम उचित उपसमूच्यों (proper subsets) में विभाजित कर सकते हैं (ऐसे कि एक संख्या एक ही उपसमूच्य में हो) ताकि हर उपसमूच्य में आने वाली संख्याओं का योग k हो। कितनी संख्याएँ अच्छी हैं?

23. ऐसा सबसे बड़ा पूर्णांक n कौनसा है जिससे कि

$$\frac{a^2}{\frac{b}{29}+\frac{c}{31}}+\frac{b^2}{\frac{c}{29}+\frac{a}{31}}+\frac{c^2}{\frac{a}{29}+\frac{b}{31}} \geq n(a+b+c)$$

सभी धनात्मक वास्तविक संख्याओं a, b, c के लिए सच हो?

24. अगर N अलग अलग आकार के त्रिभुज (मतलब असमरूप त्रिभुज) हैं जिनके सभी कोण (डिग्री में) पूर्णांक हैं तो $[N/100]$ का मान क्या होगा?

25. मान लो कि T सबसे छोटा धनात्मक पूर्णांक है जिसका $11, 13$ व 15 से विभाजन करने पर शेष क्रमशः समूच्य $\{7, 8, 9\}, \{1, 2, 3\}$ व $\{4, 5, 6\}$ में है। तो फिर T के अंकों के वर्ग का योग क्या होगा?

26. 11×11 की शतरंज की बिसात से 60 इकाई वर्ग कितनी तरह से चुन सकते हैं कि चुने हुए वर्गों की कोई भी भुजा साझा ना हो?

रफ़ कार्य के लिए जगह

27. What is the number of ways in which one can colour the squares of a 4×4 chessboard with colours red and blue such that each row as well as each column has exactly two red squares and two blue squares?

28. Let N be the number of ways of distributing 8 chocolates of different brands among 3 children such that each child gets at least one chocolate, and no two children get the same number of chocolates. Find the sum of the digits of N.

29. Let D be an interior point of the side BC of a triangle ABC. Let I_1 and I_2 be the incentres of triangles ABD and ACD respectively. Let AI_1 and AI_2 meet BC in E and F respectively. If $\angle BI_1E = 60^\circ$, what is the measure of $\angle CI_2F$ in degrees?

30. Let $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be a polynomial in which a_i is a non-negative integer for each $i \in \{0, 1, 2, 3, \cdots, n\}$. If $P(1) = 4$ and $P(5) = 136$, what is the value of $P(3)$?

SPACE FOR ROUGH WORK

27. 4×4 की शतरंज की बिसात के हर एक वर्ग को लाल या नीले में रंगना है, ऐसे कितने तरीक़े होंगे कि हर एक बेड़ी पंक्ति और हर एक खड़ी पंक्ति में दो नीले व दो लाल वर्ग हों?

28. मान लो कि अलग-अलग कम्पनियों की 8 चोकलेट को तीन बच्चों में N तरीक़ों से बाँटा जा सकता है जिससे कि हर बच्चे को कम से कम एक चोकलेट मिले और किन्हीं भी दो बच्चों को बराबर की संख्या में चोकलेट ना मिले। N के अंकों का योग कितना होगा?

29. मान लो कि D एक त्रिभुज ABC की भुजा BC का आंतरिक बिंदु है। मान लो कि I_1 व I_2 क्रमशः त्रिभुज ABC व त्रिभुज ACD के अंतःकेंद्र हैं। मान लो कि रेखा AI_1 व रेखा AI_2 रेखा BC से क्रमशः E व F में मिलती है। अगर $\angle BI_1E = 60°$, तो डिग्री में $\angle CI_2F$ का मान क्या होगा?

30. यदि $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ एक बहुपद है जहाँ a_i अऋणात्मक पूर्णांक हैं, व $P(1) = 4$ व $P(5) = 136$ तो $P(3)$ का मान क्या होगा?

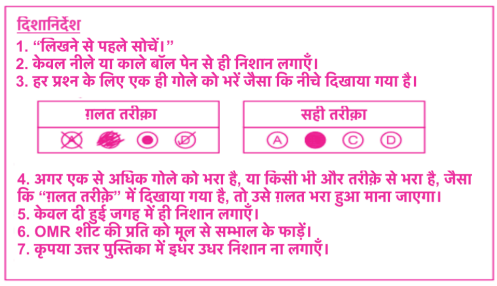
रफ़ कार्य के लिए जगह

MTA-IAPT पूर्व-क्षेत्रीय गणित ओलम्पियाड (PRMO), 2018

दिनांक: अगस्त 19, 2018 समय: 10:00 प्रातः से 01:00 दोपहर
कुल प्रश्न: 30 अधिकतम अंक: 102

दिशानिर्देश

1. मोबाइल फ़ोन, स्मार्टफ़ोन, टैब्लेट-कम्प्यूटर, कैलकुलेटर, प्रोग्रामेबल घड़ी आदि का प्रयोग पूर्णतः वर्जित है। केवल साधारण पेन व पेंसिल ही परीक्षा केंद्र में ले जा सकते हैं।
2. उत्तरों की जाँच मशीनों से होगी। OMR शीट पर बुलबुलों को काली पेंसिल या फिर नीले या काले पेन से पूरी तरह भरें। उत्तर सुनिश्चित करने के बाद ही बुलबुलों को भरें; वरना मिटाने पर OMR शीट ख़राब हो सकती है जिससे कि मशीन उत्तर नहीं पढ़ पाएगी।
3. OMR शीट पर आपके द्वारा दिए जाने वाले नाम, ईमेल-आइ.डी., व जन्मतिथि के ज़रिए लॉगिन करके आप PRMO का परिणाम देख सकेंगे।
4. अपूर्ण/ग़लत या ध्यान से नहीं भारी गई जानकारी की वजह से आप इस परीक्षा में अमान्य घोषित हो सकते हैं।
5. हर प्रश्न का उत्तर 1 या 2 अंकों की संख्या है। पहले पृष्ठ के पहले चित्र में बुलबुले भरने के सही व ग़लत तरीक़े, निर्देशों के साथ दर्शाए गए हैं। दूसरे चित्र में दर्शाया गया है कि 1-अंक व 2-अंक की संख्या को कैसे चिन्हित करेंगे।


दिशानिर्देश
1. "लिखने से पहले सोचें।"
2. केवल नीले या काले बॉल पेन से ही निशान लगाएँ।
3. हर प्रश्न के लिए एक ही गोले को भरें जैसा कि नीचे दिखाया गया है।
ग़लत तरीक़ा
सही तरीक़ा
A B C D
4. अगर एक से अधिक गोले को भरा है, या किसी भी और तरीक़े से भरा है, जैसा कि "ग़लत तरीक़े" में दिखाया गया है, तो उसे ग़लत भरा हुआ माना जाएगा।
5. केवल दी हुई जगह में ही निशान लगाएँ।
6. OMR शीट की प्रति को मूल से सम्भाल के फाड़ें।
7. कृपया उत्तर पुस्तिका में इधर उधर निशान ना लगाएँ।

6. OMR शीट पर लिखा गया उत्तर किसी काम का नहीं है। भरे हुए बुलबुले ही आपका उत्तर माने जाएँगे।
7. प्रश्न 1 से प्रश्न 6 तक हर सही उत्तर पर 2 अंक (marks) मिलेंगे, प्रश्न 7 से प्रश्न 21 तक हर सही उत्तर पर 3 अंक (marks) मिलेंगे व प्रश्न 22 से प्रश्न 30 तक हर सही उत्तर पर 5 अंक मिलेंगे।
8. सभी प्रश्न अनिवार्य हैं।
9. कोई ऋणात्मक अंक (negative marks) नहीं हैं।
10. सभी रफ़ कार्य प्रश्नों के नीचे दी गई ख़ाली जगह में करें। आपके पास प्रश्न-पत्र के अंत में भी रफ़ कार्य के लिए ख़ाली जगह है।
11. परीक्षा के बाद परोक्षार्थियों के लिए चिन्हित OMR शीट की प्रति आप ले जा सकते हैं।
12. अपनी OMR शीट की प्रति इस बार के ओलम्पियाड ख़त्म होने तक सम्भाल कर रखें। इसकी बाद में ज़रूरत पड़ेगी।
13. परीक्षा के उपरांत आप आप प्रश्न पत्र अपने साथ ले जा सकते हैं।

MTA Pre-Regional Mathematical Olympiad (PRMO),2019

Date: August 11, 2019 **Time: 10 AM to 1 PM**

Number of Questions 30: Max Marks: 102

INSTRUCTIONS

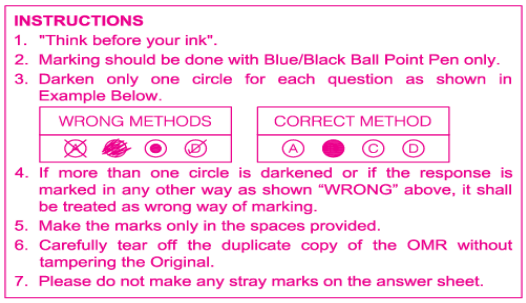
1. Use of mobile phones, smartphones, ipads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a black pencil or a black or blue ball pen. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer.

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your PRMO score.

4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.


INSTRUCTIONS
1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.
WRONG METHODS
CORRECT METHOD
A
C
D
4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.

6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be cosidered as your final answer.

7. Questions 1 to 6 carry 2 marks each; questions 7 to 21 carry 3 marks each; questions 22 to 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

(हिन्दी दिशानिर्देश पुस्तिका के अंतिम पृष्ठ पर हैं)

1. From a square with sides of length 5, triangular pieces from the four corners are removed to form a regular octagon. Find the area **removed** to the nearest integer?

2. Let $f(x) = x^2 + ax + b$. If for all nonzero real x

$$f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

and the roots of $f(x) = 0$ are integers, what is the value of $a^2 + b^2$?

3. Let x_1 be a positive real number and for every integer $n \geq 1$ let $x_{n+1} = 1 + x_1 x_2 \ldots x_{n-1} x_n$. If $x_5 = 43$, what is the sum of digits of the largest prime factor of x_6?

4. An ant leaves the anthill for its morning exercise. It walks 4 feet east and then makes a 160° turn to the right and walks 4 more feet. It then makes another 160° turn to the right and walks 4 more feet. If the ant continues this pattern until it reaches the anthill again, what is the distance in feet it would have walked?

5. Five persons wearing badges with numbers $1, 2, 3, 4, 5$ are seated on 5 chairs around a circular table. In how many ways can they be seated so that no two persons whose badges have consecutive numbers are seated next to each other? (Two arrangements obtained by rotation around the table are considered different.)

6. Let $\overline{abc}$ be a three digit number with nonzero digits such that $a^2 + b^2 = c^2$. What is the largest possible prime factor of $\overline{abc}$?

SPACE FOR ROUGH WORK

1. एक वर्ग जिसकी भुजाओं की लम्बाई 5 है, उसके चारों कोनों से त्रिभुजाकार टुकड़े काट कर एक नियमित अष्टभुज बनाया जाता है। कितना क्षेत्रफल हटाया गया है, उसका मान सबसे करीबी पूर्णांक तक ज्ञात करो।

2. मान लो कि $f(x) = x^2 + ax + b$ है। यदि सभी अशून्य वास्तविक संख्या x के लिए

$$f\left(x + \frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

और $f(x) = 0$ के सभी हल पूर्णांक हैं, तो $a^2 + b^2$ का मान क्या है?

3. मान लो कि x_1 एक धनात्मक वास्तविक संख्या है और सभी $n \geq 1$ पूर्णांकों के लिए $x_{n+1} = 1 + x_1 x_2 \ldots x_{n-1} x_n$ है। अगर $x_5 = 43$ तो x_6 के सबसे बड़े अभाज्य गुणनखंड के अंकों (digits) का योग ज्ञात करो।

4. एक चींटी अपनी बांबी से सुबह के व्यायाम के लिए निकलती है। वह 4 फुट पूरब की ओर चलती है, फिर 160° दाएँ ओर मुड़ कर 4 फुट और चलती है। फिर वह एक बार और 160° दाएँ ओर मुड़ कर 4 फुट चलती है। अगर चींटी इसी क्रम में चलती रहती तो वापिस अपनी बांबी तक पहुंचने में उसने कुल कितनी दूरी (फुट में) चली होती?

5. पाँच बिल्ले, जिनपर $1, 2, 3, 4, 5$ लिखा है, पहने हुए पाँच लोग एक गोल मेज के चारों तरफ पाँच कुर्सियों पर बैठे हैं। वह ऐसी कितनी तरह से बैठ सकते हैं, जिससे कि कोई भी दो लोग जिनके बिल्ले पर लिखी संख्याएँ क्रमागत हों, वह अगल-बगल ना बैठे हों? (एक बैठने का तरीका जो किसी दूसरे तरीके को घुमा देने से मिलता हो, उसे दूसरे तरीके से भिन्न माना जाएगा।)

6. मान लो कि $\overline{abc}$ तीन अंकों की ऐसी संख्या है, जिसके अंक अशून्य हैं व $a^2 + b^2 = c^2$ है। $\overline{abc}$ का सबसे बड़ा अभाज्य गुणनखंड क्या संभव है?

रफ़ कार्य के लिए जगह

7. On a clock, there are two instants between 12 noon and 1 PM, when the hour hand and the minute hand are at right angles. The difference *in minutes* between these two instants is written as $a + \frac{b}{c}$, where a, b, c are positive integers, with $b < c$ and b/c in the reduced form. What is the value of $a + b + c$?

8. How many positive integers n are there such that $3 \leq n \leq 100$ and $x^{2^n} + x + 1$ is divisible by $x^2 + x + 1$?

9. Let the rational number p/q be closest to but not equal to $22/7$ among all rational numbers with denominator < 100. What is the value of $p - 3q$?

10. Let ABC be a triangle and let Ω be its circumcircle. The internal bisectors of angles A, B and C intersect Ω at A_1, B_1, and C_1, respectively, and the internal bisectors of angles A_1, B_1 and C_1 of the triangle $A_1B_1C_1$ intersect Ω at A_2, B_2 and C_2, respectively. If the smallest angle of triangle ABC is $40°$, what is the magnitude of the smallest angle of triangle $A_2B_2C_2$ in degrees?

11. How many distinct triangles ABC are there, up to similarity, such that the magnitudes of angles A, B and C in degrees are positive integers and satisfy

$$\cos A \cos B + \sin A \sin B \sin kC = 1$$

for some positive integer k, where kC does not exceed $360°$?

SPACE FOR ROUGH WORK

7. एक घड़ी में दोपहर 12 बजे और दोपहर 1 बजे के बीच में दो बार ऐसा होता है जब घंटे की सूंई और मिनट की सूंई एक दूसरे के लम्ब होती है। इन दो समय के बीच में मिनटों में अंतर $a + \frac{b}{c}$ की तरह लिखें, जहाँ a, b, c धनात्मक पूर्णांक हैं, $b < c$, व b/c भिन्न का न्यूनतम रूप (reduced form या simplest form) है। $a + b + c$ का मान क्या है?

8. ऐसे कितने धनात्मक पूर्णांक n हैं जिनके लिए $3 \leq n \leq 100$ व संख्या $x^{2^n} + x + 1$ संख्या $x^2 + x + 1$ से भाज्य है?

9. मान लो कि p/q ऐसा भिन्न है जो ऐसे भिन्न जिनका हर (denominator) < 100 हो और जो $22/7$ से अलग हों, उनमें $22/7$ के सबसे करीब पड़ता है। तो $p - 3q$ का मान क्या होगा?

10. मान लो कि ABC एक त्रिभुज है और Ω उसका बहिवृत्त है। कोण A, B व C के अन्तःसमद्विभाजक Ω से क्रमशः A_1, B_1 व C_1 में मिलते हैं और त्रिभुज $A_1B_1C_1$ के कोण A_1, B_1 व C_1 के अन्तःसमद्विभाजक Ω से क्रमशः A_2, B_2, C_2 में मिलते हैं। अगर त्रिभुज ABC का सबसे छोटा कोण 40° है तो त्रिभुज $A_2B_2C_2$ के सबसे छोटे कोण का मान डिग्री (degree) में क्या होगा?

11. ऐसे कितने अलग-अलग (मतलब असमरूप) त्रिभुज ABC हैं जिनमें कोण A, B व C का डिग्री में मान धनात्मक पूर्णांक है, और जो

$$\cos A \cos B + \sin A \sin B \sin kC = 1$$

समीकरण को किसी धनात्मक पूर्णांक k के लिए संतुष्ट करते हैं, जहाँ kC का मान 360° से ज्यादा नहीं है?

रफ़ कार्य के लिए जगह

12. A natural number $k > 1$ is called *good* if there exist natural numbers

$$a_1 < a_2 < \cdots < a_k$$

such that

$$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \ldots + \frac{1}{\sqrt{a_k}} = 1.$$

Let $f(n)$ be the sum of the first n *good* numbers, $n \geq 1$. Find the sum of all values of n for which $f(n+5)/f(n)$ is an integer.

13. Each of the numbers $x_1, x_2, \ldots, x_{101}$ is ± 1. What is the smallest positive value of $\sum_{1 \leq i < j \leq 101} x_i x_j$?

14. Find the smallest positive integer $n \geq 10$ such that $n + 6$ is a prime and $9n + 7$ is a perfect square.

15. In how many ways can a pair of parallel diagonals of a regular polygon of 10 sides be selected?

16. A pen costs Rs. 13 and a note book costs Rs. 17. A school spends exactly Rs. 10000 in the year 2017-18 to buy x pens and y note books such that x and y are as close as possible (i.e., $|x - y|$ is minimum). Next year, in 2018-19, the school spends a little more than Rs. 10000 and buys y pens and x note books. How much **more** did the school pay?

SPACE FOR ROUGH WORK

12. किसी प्राकृतिक संख्या $k > 1$ को हम अच्छा कहेंगे अगर ऐसी प्राकृतिक संख्याएँ

$$a_1 < a_2 < \cdots < a_k$$

मौजूद हों जिससे कि

$$\frac{1}{\sqrt{a_1}} + \frac{1}{\sqrt{a_2}} + \cdots + \frac{1}{\sqrt{a_k}} = 1$$

हो। मान लो कि $n \geq 1$ के लिए $f(n)$ पहली n अच्छी संख्याओं का योग है। ऐसी सभी संख्याओं n का योग ज्ञात करो जिनके लिए $f(n+5)/f(n)$ एक पूर्णांक है।

13. $x_1, x_2, \ldots, x_{101}$ में हर एक संख्या ± 1 है। $\displaystyle\sum_{1 \leq i < j \leq 101} x_i x_j$ का न्यूनतम धनात्मक मान क्या है?

14. ऐसा सबसे छोटा धनात्मक पूर्णांक $n \geq 10$ ज्ञात करो जहाँ $n+6$ अभाज्य हो और $9n+7$ एक पूर्ण वर्ग हो।

15. नियमित दसभुज के विकर्णों के ऐसे कितने जोड़े चुने जा सकते हैं जिसमें कि दोनों विकर्ण समानांतर हों?

16. एक पेन का मूल्य रु॰ 13 है व एक कापी का मूल्य रु॰ 17 है। एक स्कूल 2017-18 में x पेन और y कापियाँ खरीदने में ठीक रु॰ 10000 खर्च करता है, ऐसे कि x व y जितना हो सकें उतना करीब हों (माने कि $|x-y|$ न्यूनतम हो)। अगले साल, 2018-19 में, स्कूल रु॰ 10000 से थोड़ा ज्यादा खर्च करता है, और y पेन व x कापियाँ खरीदता है। स्कूल ने कितना ज्यादा खर्च किया?

रफ़ कार्य के लिए जगह

17. Find the number of ordered triples (a, b, c) of positive integers such that $30a+50b+70c \leq 343$.

18. How many ordered pairs (a, b) of positive integers with $a < b$ and $100 \leq a, b \leq 1000$ satisfy $\gcd(a, b) : \text{lcm}(a, b) = 1 : 495$?

19. Let AB be a diameter of a circle and let C be a point on the segment AB such that $AC : CB = 6 : 7$. Let D be a point on the circle such that DC is perpendicular to AB. Let DE be the diameter through D. If $[XYZ]$ denotes the area of the triangle XYZ, find $[ABD]/[CDE]$ to the nearest integer.

20. Consider the set E of all natural numbers n such that when divided by 11,12, 13, respectively, the remainders, in that order, are distinct prime numbers in an arithmetic progression. If N is the largest number in E, find the sum of digits of N.

21. Consider the set $E = \{5, 6, 7, 8, 9\}$. For any partition $\{A, B\}$ of E, with both A and B non-empty, consider the number obtained by adding the product of elements of A to the product of elements of B. Let N be the largest prime number among these numbers. Find the sum of the digits of N.

SPACE FOR ROUGH WORK

17. धनात्मक पूर्णांकों के ऐसे कितने क्रमवार समुच्चय (a, b, c) हैं जिनके लिए $30a + 50b + 70c \leq 343$ है?

18. धनात्मक पूर्णांकों के ऐसे कितने क्रमवार जोड़े (a, b) हैं जहाँ $a < b$ व $100 \leq a, b \leq 1000$ और जिनके लिए $\gcd(a, b) : \text{lcm}(a, b) = 1 : 495$? (यहाँ gcd = म॰स॰ = महत्तम समापवर्तक, lcm = ल॰स॰ = लघुत्तम समापवर्त्य।)

19. मान लो कि AB एक वृत्त का व्यास है और मान लो कि C रेखाखंड AB पर ऐसा बिंदु है जिससे कि $AC : CB = 6 : 7$ है। मान लो कि D वृत्त पर एक ऐसा बिंदु है कि DC खंड AB पर लंब है। मान लो कि DE बिंदु D से गुजरता हुआ वृत्त का व्यास है। अगर $[XYZ]$ से हमारा मतलब त्रिभुज XYZ का क्षेत्रफल है तो $[ABD]/[CDE]$ का मान सबसे करीबी पूर्णांक तक ज्ञात करो।

20. मान लो कि E ऐसी प्राकृतिक संख्याओं n का समुच्चय है जिन्हें 11, 12, 13 से भाग देने पर मिलने वाले शेष ऐसी अलग-अलग अभाज्य संख्याएँ हैं जो इसी क्रम में समान्तर श्रेणी में हैं। अगर N समुच्चय E की सबसे बड़ी संख्या है तो N के अंकों का योग ज्ञात करो।

21. समुच्चय $E = \{5, 6, 7, 8, 9\}$ को लो। E के किसी भी विभाजन $\{A, B\}$ के लिए, जहाँ A और B दोनों अरिक्त हैं, A के सदस्यों के गुणनफल का B के सदस्यों के गुणनफल से योग लेने पर मिलने वाली संख्या को लो। N इन सभी संख्याओं में सबसे बड़ा अभाज्य है। N के अंकों (digits) का योग ज्ञात करो।

रफ़ कार्य के लिए जगह

22. What is the greatest integer not exceeding the sum $\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}$?

23. Let $ABCD$ be a convex cyclic quadrilateral. Suppose P is a point in the plane of the quadrilateral such that the sum of its distances from the vertices of $ABCD$ is the least. If

$$\{PA, PB, PC, PD\} = \{3, 4, 6, 8\},$$

what is the maximum possible area of $ABCD$?

24. A $1 \times n$ rectangle ($n \geq 1$) is divided into n unit (1×1) squares. Each square of this rectangle is coloured red, blue or green. Let $f(n)$ be the number of colourings of the rectangle in which there are an even number of red squares. What is the largest prime factor of $f(9)/f(3)$? (The number of red squares can be zero.)

25. A village has a circular wall around it, and the wall has four gates pointing north, south, east and west. A tree stands outside the village, 16 m north of the north gate, and it can be *just* seen appearing on the horizon from a point 48 m east of the south gate. What is the diameter, in meters, of the wall that surrounds the village?

26. Positive integers x, y, z satisfy $xy + z = 160$. Compute the smallest possible value of $x + yz$.

SPACE FOR ROUGH WORK

22. ऐसा सबसे बड़ा पूर्णांक ज्ञात करो जो कि योग $\sum_{n=1}^{1599} \frac{1}{\sqrt{n}}$ से बड़ा ना हो।

23. मान लो कि $ABCD$ एक उत्तल (convex) चक्रीय (cyclic) चतुर्भुज है। मान लो कि P चतुर्भुज के तल में ऐसा बिंदु है जिसकी चतुर्भुज $ABCD$ के शीर्षों से दूरियों का योग न्यूनतम है। अगर

$$\{PA, PB, PC, PD\} = \{3, 4, 6, 8\}$$

तो $ABCD$ का अधिकतम संभव क्षेत्रफल कितना है?

24. एक $1 \times n$ आयत ($n \geq 1$) को n इकाई (1×1) वर्गों में बाँट देते हैं। इस आयत के हर वर्ग को लाल, नीला या हरा रंग देते हैं। मान लो कि $f(n)$ वह संख्या है जितने तरीकों से ऐसे रंग सकते हैं कि लाल रंग के आयतों की संख्या सम हो। $f(9)/f(3)$ का सबसे बड़ा अभाज्य गुणनखंड क्या है? (लाल वर्गों की संख्या शून्य हो सकती है।)

25. एक गाँव के चारों तरफ एक गोल दीवार है जिसमें पूरब, पश्चिम, उत्तर व दक्षिण की तरफ चार द्वार हैं। गाँव के बाहर लगा एक पेड़ उत्तर द्वार के 16 मी॰ उत्तर में है और वह दक्षिण द्वार के 48 मी॰ पूरब स्थित बिन्दु से बस क्षितिज पर उभरता हुआ थोड़ा सा दिखाई देता है। गाँव को घेरती हुई दीवार का मीटर में व्यास क्या है?

26. धनात्मक पूर्णांक x, y, z समीकरण $xy + z = 160$ को संतुष्ट करते हैं। $x + yz$ का न्यूनतम संभव मान ज्ञात करो।

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27. We will say that a rearrangement of the letters of a word has *no fixed letters* if, when the rearrangement is placed directly below the word, no column has the same letter repeated. For instance, $H\,B\,R\,A\,T\,A$ is a rearrangement with no fixed letters of $B\,H\,A\,R\,A\,T$. How many distinguishable rearrangements with no fixed letters does $B\,H\,A\,R\,A\,T$ have? (The two As are considered identical.)

28. Let ABC be a triangle with sides 51, 52, 53. Let Ω denote the incircle of $\triangle ABC$. Draw tangents to Ω which are parallel to the sides of ABC. Let r_1, r_2, r_3 be the inradii of the three corner triangles so formed. Find the largest integer that does not exceed $r_1 + r_2 + r_3$.

29. In a triangle ABC, the median AD (with D on BC) and the angle bisector BE (with E on AC) are perpendicular to each other. If $AD = 7$ and $BE = 9$, find the integer nearest to the area of triangle ABC.

30. Let E denote the set of all natural numbers n such that $3 < n < 100$ and the set $\{1, 2, 3, \ldots, n\}$ can be partitioned in to 3 subsets with equal sums. Find the number of elements of E.

SPACE FOR ROUGH WORK

27. हम कहेंगे कि किसी शब्द के अक्षरों के क्रमचय में कोई स्थिर अक्षर नहीं है अगर, जब हम क्रमचय को शब्द के ठीक नीचे लिखें, तो किसी भी स्तंभ में एक ही अक्षर ऊपर नीचे दोनों जगह नहीं होगा। जैसे कि शब्द $B\ H\ A\ R\ A\ T$ के क्रमचय $H\ B\ R\ A\ T\ A$ में कोई स्थिर अक्षर नहीं है। $B\ H\ A\ R\ A\ T$ के अलग-अलग ऐसे कितने क्रमचय हैं जिनमें कोई स्थिर अक्षर नहीं है? (यहाँ मानेंगे कि दोनों A में कोई भेद नहीं है।)

28. मान लो कि ABC एक त्रिभुज है जिसकी भुजाओं का मान 51, 52, 53 है। मान लो कि Ω त्रिभुज $\triangle ABC$ का अंतःवृत्त है। Ω के वह स्पर्शक बनाओ जो कि ABC की भुजाओं के समानांतर हैं। मान लो कि कोनों पर बने तीन त्रिभुजों का अंतःव्यास r_1, r_2, r_3 है। ऐसा अधिकतम पूर्णांक कौनसा होगा जो $r_1 + r_2 + r_3$ के मान से ज्यादा न हो?

29. एक त्रिभुज ABC में मध्यिका AD (जहाँ D भुजा BC पर है) और कोण-समद्विभाजक BE (जहाँ E भुजा AC पर है) लम्ब हैं। अगर $AD = 7$ व $BE = 9$ तो ABC के क्षेत्रफल के सबसे करीबी पूर्णांक का मान ज्ञात करो।

30. मान लो कि E ऐसी प्राकृतिक संख्यों n का समुच्चय है जिनके लिए $3 < n < 100$ और जिनके लिए समुच्चय $\{1, 2, 3, \ldots, n\}$ को ऐसे तीन उपसमुच्चयों में विभाजित किया जा सकता है जिनका योग बराबर है। E के सदस्यों की संख्या ज्ञात करो।

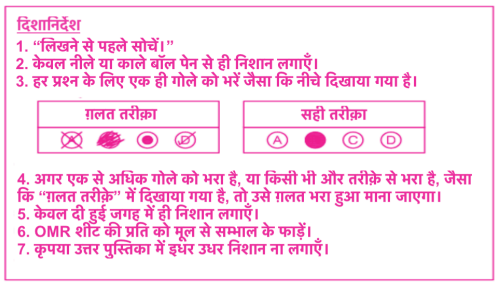
रफ़ कार्य के लिए जगह

MTA-IAPT पूर्व-क्षेत्रीय गणित ओलम्पियाड (PRMO), 2019

दिनांक: अगस्त 11, 2019 समय: 10:00 प्रातः से 01:00 दोपहर
कुल प्रश्न: 30 अधिकतम अंक: 102

दिशानिर्देश

1. मोबाइल फ़ोन, स्मार्टफ़ोन, टैब्लेट-कम्प्यूटर, कैलकुलेटर, प्रोग्रामेबल घड़ी आदि का प्रयोग पूर्णतः वर्जित है। केवल साधारण पेन व पेंसिल ही परीक्षा केंद्र में ले जा सकते हैं।
2. उत्तरों की जाँच मशीनों से होगी। OMR शीट पर बुलबुलों को काली पेंसिल या फिर नीले या काले पेन से पूरी तरह भरें। उत्तर सुनिश्चित करने के बाद ही बुलबुलों को भरें; वरना मिटाने पर OMR शीट ख़राब हो सकती है जिससे कि मशीन उत्तर नहीं पढ़ पाएगी।
3. OMR शीट पर आपके द्वारा दिए जाने वाले नाम, ईमेल-आईडी व जन्मतिथि के ज़रिए लॉगिन करके आप PRMO का परिणाम देख सकेंगे।
4. अपूर्ण/ग़लत या ध्यान से नहीं भरी गई जानकारी की वजह से आप इस परीक्षा में अमान्य घोषित हो सकते हैं।
5. हर प्रश्न का उत्तर 1 या 2 अंकों की संख्या है। पहले पृष्ठ के पहले चित्र में बुलबुले भरने के सही व ग़लत तरीक़े, निर्देशों के साथ दर्शाए गए हैं। दूसरे चित्र में दर्शाया गया है कि 1-अंक व 2-अंक की संख्या को कैसे चिन्हित करेंगे।


दिशानिर्देश
1. "लिखने से पहले सोचें।"
2. केवल नीले या काले बॉल पेन से ही निशान लगाएँ।
3. हर प्रश्न के लिए एक ही गोले को भरें जैसा कि नीचे दिखाया गया है।
ग़लत तरीक़ा
सही तरीक़ा
A C D
4. अगर एक से अधिक गोले को भरा है, या किसी भी और तरीक़े से भरा है, जैसा कि "ग़लत तरीक़े" में दिखाया गया है, तो उसे ग़लत भरा हुआ माना जाएगा।
5. केवल दी हुई जगह में ही निशान लगाएँ।
6. OMR शीट की प्रति को मूल से सम्भाल के फाड़ें।
7. कृपया उत्तर पुस्तिका में इधर उधर निशान ना लगाएँ।

6. OMR शीट पर लिखा गया उत्तर किसी काम का नहीं है। भरे हुए बुलबुले ही आपका उत्तर माने जाएँगे।
7. प्रश्न 1 से प्रश्न 6 तक हर सही उत्तर पर 2 अंक (marks) मिलेंगे, प्रश्न 7 से प्रश्न 21 तक हर सही उत्तर पर 3 अंक (marks) मिलेंगे व प्रश्न 22 से प्रश्न 30 तक हर सही उत्तर पर 5 अंक मिलेंगे।
8. सभी प्रश्न अनिवार्य हैं।
9. कोई ऋणात्मक अंक (negative marks) नहीं हैं।
10. सभी रफ़ कार्य प्रश्नों के नीचे दी गई ख़ाली जगह में करें। आपके पास प्रश्न-पत्र के अंत में भी रफ़ कार्य के लिए ख़ाली जगह है।
11. परीक्षा के बाद परीक्षार्थियों के लिए चिन्हित OMR शीट की प्रति आप ले जा सकते हैं।
12. अपनी OMR शीट की प्रति इस बार के ओलम्पियाड ख़त्म होने तक सम्भाल कर रखें। इसकी बाद में ज़रूरत पड़ेगी।
13. परीक्षा के उपरांत आप अपना प्रश्न पत्र अपने साथ ले जा सकते हैं।

Code : **M01**

Roll Number: | | |

1. Let $ABCD$ be a trapezium in which $AB \parallel CD$ and $AB = 3CD$. Let E be the midpoint of the diagonal BD. If $[ABCD] = n \times [CDE]$, what is the value of n? (Here $[\Gamma]$ denotes the area of the geometrical figure Γ.)

2. A number N in base 10, is 503 in base b and 305 in base $b+2$. What is the product of the digits of N?

3. If $\sum_{k=1}^{N} \frac{2k+1}{(k^2+k)^2} = 0.9999$ then determine the value of N.

4. Let $ABCD$ be a rectangle in which $AB + BC + CD = 20$ and $AE = 9$ where E is the mid-point of the side BC. Find the area of the rectangle.

5. Find the number of integer solutions to $\Big||x| - 2020\Big| < 5$.

6. What is the least positive integer by which $2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7$ should be multiplied so that, the product is a perfect square ?

7. Let ABC be a triangle with $AB = AC$. Let D be a point on the segment BC such that $BD = 48\frac{1}{61}$ and $DC = 61$. Let E be a point on AD such that CE is perpendicular to AD and $DE = 11$. Find AE.

8. A 5-digit number (in base 10) has digits k, $k+1$, $k+2$, $3k$, $k+3$ in that order, from left to right. If this number is m^2 for some natural number m, find the sum of the digits of m.

SPACE FOR ROUGH WORK

9. Let ABC be a triangle with $AB = 5$, $AC = 4$, $BC = 6$. The internal angle bisector of C intersects the side AB at D. Points M and N are taken on sides BC and AC, respectively, such that $DM \parallel AC$ and $DN \parallel BC$. If $(MN)^2 = \frac{p}{q}$ where p and q are relatively prime positive integers then what is the sum of the digits of $|p - q|$?

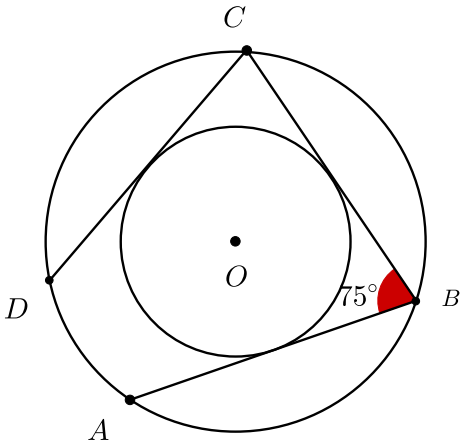
10. Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores? (The median of a set of scores is the middlemost score when the data is arranged in increasing order. It is exactly the middle score when there are an odd number of scores and it is the average of the two middle scores when there are an even number of scores.)

11. Let $X = \{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5\}$ and

    $S = \{(a, b) \in X \times X : x^2 + ax + b$ and $x^3 + bx + a$ have at least a common real zero$\}$.

    How many elements are there in S?

12. Given a pair of concentric circles, chords $AB, BC, CD, \ldots$ of the outer circle are drawn such that they all touch the inner circle. If $\angle ABC = 75°$, how many chords can be drawn before returning to the starting point?



SPACE FOR ROUGH WORK

13. Find the sum of all positive integers n for which $|2^n + 5^n - 65|$ is a perfect square.

14. The product $55 \times 60 \times 65$ is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

15. Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the other. How many arrangements are possible?

16. The sides x and y of a scalene triangle satisfy $x + \frac{2\Delta}{x} = y + \frac{2\Delta}{y}$, where Δ is the area of the triangle. If $x = 60, y = 63$, what is the length of the largest side of the triangle?

SPACE FOR ROUGH WORK

17. How many two digit numbers have exactly 4 positive factors? (Here 1 and the number n are also considered as factors of n.)

18. If

$$\sum_{k=1}^{40}\left(\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\ \right)=a+\frac{b}{c}$$

where $a,b,c\in\mathbb{N}, b<c, \gcd(b,c)=1$, then what is the value of $a+b$?

19. Let $ABCD$ be a parallelogram . Let E and F be midpoints of AB and BC respectively. The lines EC and FD intersect in P and form four triangles APB, BPC, CPD and DPA. If the area of the parallelogram is 100 sq. units, what is the maximum area in sq. units of a triangle among these four triangles?

20. A group of women working together at the same rate can build a wall in 45 hours. When the work started, all the women did not start working together. They joined the work over a period of time, one by one, at equal intervals. Once at work, each one stayed till the work was complete. If the first woman worked 5 times as many hours as the last woman, for how many hours did the first woman work?

21. A total fixed amount of N thousand rupees is given to three persons A, B, C, every year, each being given an amount proportional to her age. In the first year, A got half the total amount. When the sixth payment was made, A got six-seventh of the amount that she had in the first year; B got Rs 1000 less than that she had in the first year; and C got twice of that she had in the first year. Find N.

SPACE FOR ROUGH WORK

22. In triangle ABC, let P and R be the feet of the perpendiculars from A onto the external and internal bisectors of $\angle ABC$, respectively; and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of $\angle ACB$, respectively. If $PQ = 7$, $QR = 6$ and $RS = 8$, what is the area of triangle ABC?

23. The incircle Γ of a scalene triangle ABC touches BC at D, CA at E and AB at F. Let r_A be the radius of the circle inside ABC which is tangent to Γ and the sides AB and AC. Define r_B and r_C similarly. If $r_A = 16$, $r_B = 25$ and $r_C = 36$, determine the radius of Γ.

24. A light source at the point $(0, 16)$ in the coordinate plane casts light in all directions. A disc (a circle along with its interior) of radius 2 with center at $(6, 10)$ casts a shadow on the X axis. The length of the shadow can be written in the form $m\sqrt{n}$ where m, n are positive integers and n is square-free. Find $m + n$.

25. For a positive integer n, let $\langle n \rangle$ denote the perfect square integer closest to n. For example, $\langle 74 \rangle = 81, \langle 18 \rangle = 16$. If N is the smallest positive integer such that

$$\langle 91 \rangle \cdot \langle 120 \rangle \cdot \langle 143 \rangle \cdot \langle 180 \rangle \cdot \langle N \rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot N$$

find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

26. In the figure below, 4 of the 6 disks are to be colored black and 2 are to be colored white. Two colorings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same.

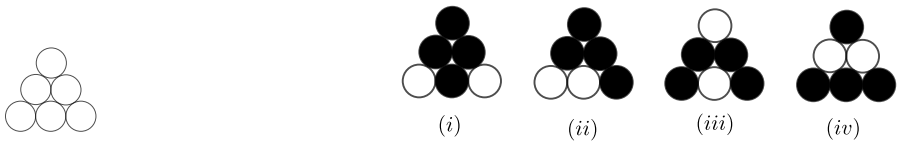


Fig 1.

There are only four such colorings for the given two colors, as shown in Figure 1. In how many ways can we color the 6 disks such that 2 are colored black, 2 are colored white, 2 are colored blue with the given identification condition?

27. A bug travels in the coordinate plane moving only along the lines that are parallel to the x axis or y axis. Let $A = (-3, 2)$ and $B(3, -2)$. Consider all possible paths of the bug from A to B of length at most 14. How many points with integer coordinates lie on at least one of these paths?

28. A natural number n is said to be *good* if n is the sum of r consecutive positive integers, for some $r \geq 2$. Find the number of good numbers in the set $\{1, 2, \ldots, 100\}$.

29. Positive integers a, b, c satisfy $\dfrac{ab}{a-b} = c$. What is the largest possible value of $a+b+c$ not exceeding 99?

30. Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, $a + 1$ divides $b - 1$ and b divides $a^2 + a + 2$.

SPACE FOR ROUGH WORK

ROUGH WORK

IOQM 2021-22 Part A

1. Three parallel lines L_1, L_2, L_3 are drawn in the plane such that the perpendicular distance between L_1 and L_2 is 3 and the perpendicular distance between L_2 and L_3 is also 3. A square $ABCD$ is constructed such that A lies on L_1, B lies on L_3 and C lies on L_2. Find the area of the square.

2. Ria writes down the numbers $1, 2, \ldots, 101$ in red and blue pens. The largest blue number is equal to the number of numbers written in blue and the smallest red number is equal to half the number of numbers written in red. How many numbers did Ria write with red pen?

3. Consider the set $\mathcal{T}$ of all triangles whose sides are distinct prime numbers which are also in arithmetic progression. Let $\triangle \in \mathcal{T}$ be the triangle with the least perimeter. If a° is the largest angle of $\triangle$ and if L is its perimeter, determine the value of $\frac{a}{L}$.

4. Consider the set of all 6-digit numbers consisting of only 3 digits, a, b, c, where a, b, c are distinct. Suppose the sum of all these numbers is 593999406. What is the largest remainder when the three digit number abc is divided by 100?

5. In parallelogram $ABCD$ the longer side is twice the shorter side. Let $XYZW$ be the quadrilateral formed by the internal bisectors of the angles of $ABCD$. If the area of $XYZW$ is 10, find the area of $ABCD$.

6. Let x, y, z be positive real numbers such that $x^2 + y^2 = 49$, $y^2 + yz + z^2 = 36$ and $x^2 + \sqrt{3}xz + z^2 = 25$. If the value of $2xy + \sqrt{3}yz + zx$ can be written as $p\sqrt{q}$ where p, q are integers and q is not divisible by square of any prime number, find $p + q$.

7. Find the number of maps $f : \{1, 2, 3\} \longrightarrow \{1, 2, 3, 4, 5\}$ such that $f(i) \leq f(j)$ whenever $i < j$.

8. For any real number t, let $\lfloor t \rfloor$ denote the largest integer $\leq t$. Suppose that N is the greatest integer such that
$$\left\lfloor \sqrt{\left\lfloor \sqrt{\lfloor \sqrt{N} \rfloor} \right\rfloor} \right\rfloor = 4$$
Find the sum of digits of N.

9. Let $P_0 = (3, 1)$ and define $P_{n+1} = (x_n, y_n)$ for $n \geq 0$ by
$$x_{n+1} = -\frac{3x_n - y_n}{2}, \qquad y_{n+1} = -\frac{x_n + y_n}{2}$$
Find the area of the quadrilateral formed by the points $P_{96}, P_{97}, P_{98}, P_{99}$.

10. Suppose that P is the polynomial of least degree with integer coefficients such that $P\left(\sqrt{7} + \sqrt{5}\right) = 2\left(\sqrt{7} - \sqrt{5}\right)$. Find $P(2)$.

11. In how many ways can four married couples sit in a merry-go-round with identical seats such that men and women occupy alternate seats and no husband seats next to his wife?

12. A 12×12 board is divided into 144 unit squares by drawing lines parallel to the sides. Two rooks placed on two unit squares are said to be non attacking if they are not in the same column or same row. Find the least number N such that if N rooks are placed on the unit squares, one rook per square, we can always find 7 rooks such that no two are attacking each other.

M1

1. A triangle ABC with $AC = 20$ is inscribed in a circle ω. A tangent t to ω is drawn through B. The distance of t from A is 25 and that from C is 16. If S denotes the area of the triangle ABC, find the largest integer not exceeding $S/20$.

2. In a parallelogram $ABCD$, a point P on the segment AB is taken such that $\dfrac{AP}{AB} = \dfrac{61}{2022}$ and a point Q on the segment AD is taken such that $\dfrac{AQ}{AD} = \dfrac{61}{2065}$. If PQ intersects AC at T, find $\dfrac{AC}{AT}$ to the nearest integer.

3. In a trapezoid $ABCD$, the internal bisector of angle A intersects the base BC (or its extension) at the point E. Inscribed in the triangle ABE is a circle touching the side AB at M and side BE at the point P. Find the angle DAE in degrees, if $AB : MP = 2$.

4. Starting with a positive integer M written on the board, Alice plays the following game: in each move, if x is the number on the board, she replaces it with $3x + 2$. Similarly, starting with a positive integer N written on the board, Bob plays the following game: in each move, if x is the number on the board, he replaces it with $2x + 27$. Given that Alice and Bob reach the same number after playing 4 moves each, find the smallest value of $M + N$.

5. Let m be the smallest positive integer such that $m^2 + (m+1)^2 + \cdots + (m+10)^2$ is the square of a positive integer n. Find $m + n$.

6. Let a, b be positive integers satisfying $a^3 - b^3 - ab = 25$. Find the largest possible value of $a^2 + b^3$.

7. Find the number of ordered pairs (a, b) such that $a, b \in \{10, 11, \cdots, 29, 30\}$ and

$$\mathrm{GCD}(a, b) + \mathrm{LCM}(a, b) = a + b.$$

8. Suppose the prime numbers p and q satisfy $q^2 + 3p = 197p^2 + q$. Write $\dfrac{q}{p}$ as $l + \dfrac{m}{n}$, where l, m, n are positive integers, $m < n$ and $\mathrm{GCD}(m, n) = 1$. Find the maximum value of $l + m + n$.

9. Two sides of an integer sided triangle have lengths 18 and x where $x < 100$. If there are exactly 35 possible integer values y such that $18, x, y$ are the sides of a non-degenerate triangle, find the number of possible integer values x can have.

10. Consider the 10-digit number $M = 9876543210$. We obtain a new 10-digit number from M according to the following rule: we can choose one or more disjoint pairs of adjacent digits in M and interchange the digits in these chosen pairs, keeping the remaining digits in their own places. For example, from $M = 9\underline{87}6\underline{54}3210$, by interchanging the 2 underlined pairs, and keeping the others in their places, we get $M_1 = 9\overline{78}6\overline{45}3210$. Note that any number of (disjoint) pairs can be interchanged. Find the number of new numbers that can be so obtained from M.

11. Let AB be a diameter of a circle ω and let C be a point on ω, different from A and B. The perpendicular from C intersects AB at D and ω at $E(\neq C)$. The circle with centre at C and radius CD intersects ω at P and Q. If the perimeter of the triangle PEQ is 24, find the length of the side PQ.

12. Given ΔABC with $\angle B = 60^\circ$ and $\angle C = 30^\circ$, let P, Q, R be points on sides BA, AC, CB respectively such that $BPQR$ is an isosceles trapezium with $PQ \parallel BR$ and $BP = QR$. Find the minimum possible value of $\dfrac{2[ABC]}{[BPQR]}$ where $[S]$ denotes the area of any polygon S.

13. Let ABC be a triangle and let D be a point on the segment BC such that $AD = BC$. Suppose $\angle CAD = x^\circ$, $\angle ABC = y^\circ$ and $\angle ACB = z^\circ$ and x, y, z are in an arithmetic progression in that order where the first term and the common difference are positive integers. Find the largest possible value of $\angle ABC$ in degrees.

14. Let x, y, z be complex numbers such that

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} = 9$$
$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} = 64$$
$$\frac{x^3}{y+z} + \frac{y^3}{z+x} + \frac{z^3}{x+y} = 488$$

If $\dfrac{x}{yz} + \dfrac{y}{zx} + \dfrac{z}{xy} = \dfrac{m}{n}$ where m, n are positive integers with $\mathrm{GCD}(m, n) = 1$, find $m+n$.

15. Let x, y be real numbers such that $xy = 1$. Let T and t be the largest and the smallest values of the expression

$$\frac{(x+y)^2 - (x-y) - 2}{(x+y)^2 + (x-y) - 2}.$$

If $T+t$ can be expressed in the form $\dfrac{m}{n}$ where m, n are nonzero integers with $\mathrm{GCD}(m, n) = 1$, find the value of $m+n$.

16. Let a, b, c be reals satisfying

$$3ab + 2 = 6b, \qquad 3bc + 2 = 5c, \qquad 3ca + 2 = 4a.$$

Let $\mathbb{Q}$ denote the set of all rational numbers. Given that the product abc can take two values $\dfrac{r}{s} \in \mathbb{Q}$ and $\dfrac{t}{u} \in \mathbb{Q}$, in lowest form, find $r+s+t+u$.

17. For a positive integer $n > 1$, let $g(n)$ denote the largest positive proper divisor of n and $f(n) = n - g(n)$. For example, $g(10) = 5, f(10) = 5$ and $g(13) = 1, f(13) = 12$. Let N be the smallest positive integer such that $f(f(f(N))) = 97$. Find the largest integer not exceeding $\sqrt{N}$.

18. Let m, n be natural numbers such that

$$m + 3n - 5 = 2\,\mathrm{LCM}(m, n) - 11\,\mathrm{GCD}(m, n).$$

Find the maximum possible value of $m+n$.

19. Consider a string of n 1's. We wish to place some $+$ signs in between so that the sum is 1000. For instance, if $n = 190$, one may put $+$ signs so as to get 11 ninety times and 1 ten times, and get the sum 1000. If a is the number of positive integers n for which it is possible to place $+$ signs so as to get the sum 1000, then find the sum of the digits of a.

20. For an integer $n \geq 3$ and a permutation $\sigma = (p_1, p_2, \ldots, p_n)$ of $\{1, 2, \ldots, n\}$, we say p_l is a *landmark point* if $2 \leq l \leq n-1$ and $(p_{l-1} - p_l)(p_{l+1} - p_l) > 0$. For example, for $n = 7$, the permutation $(2, 7, 6, 4, 5, 1, 3)$ has four landmark points: $p_2 = 7, p_4 = 4, p_5 = 5$ and $p_6 = 1$. For a given $n \geq 3$, let $L(n)$ denote the number of permutations of $\{1, 2, \ldots, n\}$ with exactly one landmark point. Find the maximum $n \geq 3$ for which $L(n)$ is a perfect square.

21. An ant is at a vertex of a cube. Every 10 minutes it moves to an adjacent vertex along an edge. If N is the number of one hour journeys that end at the starting vertex, find the sum of the squares of the digits of N.

22. A binary sequence is a sequence in which each term is equal to 0 or 1. A binary sequence is called *friendly* if each term is adjacent to at least one term that is equal to 1. For example, the sequence $0, 1, 1, 0, 0, 1, 1, 1$ is friendly. Let F_n denote the number of friendly binary sequences with n terms. Find the smallest positive integer $n \geq 2$ such that $F_n > 100$.

23. In a triangle ABC, the median AD divides $\angle BAC$ in the ratio $1 : 2$. Extend AD to E such that EB is perpendicular AB. Given that $BE = 3$, $BA = 4$, find the integer nearest to BC^2.

24. Let N be the number of ways of distributing 52 identical balls into 4 distinguishable boxes such that no box is empty and the difference between the number of balls in any two of the boxes is not a multiple of 6. If $N = 100a + b$, where a, b are positive integers less than 100, find $a + b$.

Answers

QNo	1	2	3	4	5	6	7	8	9	10	11	12
Answer	10	67	60	10	95	43	35	32	82	88	08	03

QNo	13	14	15	16	17	18	19	20	21	22	23	24
Answer	59	16	25	18	19	70	09	03	74	11	29	81

M1

Time: 3 Hours

Number of Questions: 30 **Max Marks: 100**

INSTRUCTIONS

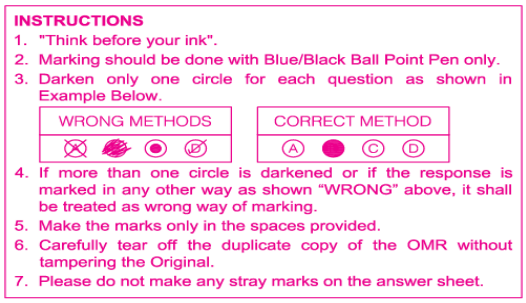
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer. .

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.

4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.


INSTRUCTIONS
1. "Think before your ink".
2. Marking should be done with Blue/Black Ball Point Pen only.
3. Darken only one circle for each question as shown in Example Below.
WRONG METHODS
CORRECT METHOD
A
C
D
4. If more than one circle is darkened or if the response is marked in any other way as shown "WRONG" above, it shall be treated as wrong way of marking.
5. Make the marks only in the spaces provided.
6. Carefully tear off the duplicate copy of the OMR without tampering the Original.
7. Please do not make any stray marks on the answer sheet.

6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 and 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

Note:

1. $\mathbb{N}$ denotes the set of all natural numbers, $1, 2, 3, \ldots$.
2. For a positive real number x, $\sqrt{x}$ denotes the positive square root of x. For example, $\sqrt{4} = +2$.
3. Unless otherwise specified, all numbers are written in base 10.

Questions

1. Let n be a positive integer such that $1 \leq n \leq 1000$. Let M_n be the number of integers in the set $X_n = \{\sqrt{4n+1}, \sqrt{4n+2}, \ldots, \sqrt{4n+1000}\}$. Let

$$a = \max\{M_n : 1 \leq n \leq 1000\}, \text{ and } b = \min\{M_n : 1 \leq n \leq 1000\}.$$

   Find $a - b$.

2. Find the number of elements in the set

$$\{(a, b) \in \mathbb{N} : 2 \leq a, b \leq 2023, \log_a(b) + 6 \log_b(a) = 5\}$$

3. Let α and β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}.$$

   Find the smallest possible value of β.

4. Let x, y be positive integers such that

$$x^4 = (x-1)(y^3 - 23) - 1.$$

   Find the maximum possible value of $x + y$.

SPACE FOR ROUGH WORK

5. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480, find the area of triangle GYZ.

6. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X.

7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

8. Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2×7 rectangle using some of these tiles.

SPACE FOR ROUGH WORK

9. Find the number of triples (a, b, c) of positive integers such that

   (a) ab is a prime;

   (b) bc is a product of two primes;

   (c) abc is not divisible by square of any prime and

   (d) $abc \leq 30$.

10. The sequence $\langle a_n \rangle_{n \geq 0}$ is defined by $a_0 = 1, a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n$, for $n \geq 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

11. A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.

12. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

SPACE FOR ROUGH WORK

13. The ex-radii of a triangle are $10\frac{1}{2}, 12$ and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p+q+r}$.

14. Let ABC be a triangle in the xy plane, where B is at the origin $(0,0)$. Let BC be produced to D such that $BC : CD = 1 : 1$, CA be produced to E such that $CA : AE = 1 : 2$ and AB be produced to F such that $AB : BF = 1 : 3$. Let $G(32, 24)$ be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK.

15. Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2. Let O be the circumcentre of triangle MAN, and P be the circumcentre of triangle MON. If $\left(\dfrac{OP}{OA}\right)^2 = \dfrac{m}{n}$ for some relatively prime positive integers m and n, find the value of $m+n$.

16. The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \le i < j < k \le 6$, has at least one red side, find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

17. Consider the set

$$\mathcal{S} = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\}$$

where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of $\mathcal{S}$, find the largest integer less than or equal to D.

18. Let $\mathcal{P}$ be a convex polygon with 50 vertices. A set $\mathcal{F}$ of diagonals of $\mathcal{P}$ is said to be *minimally friendly* if any diagonal $d \in \mathcal{F}$ intersects at most one other diagonal in $\mathcal{F}$ at a point interior to $\mathcal{P}$. Find the largest possible number of elements in a minimally friendly set $\mathcal{F}$.

19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n. Consider the set

$$A = \{n \in \mathbb{N} : P(n) \text{ is non-zero, square free and } S(n) \text{ is a proper divisor of } P(n)\}.$$

Find the maximum possible number of digits of the numbers in A.

SPACE FOR ROUGH WORK

20. For any finite non empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

$$\max(A) \times |B| = 12; \text{ and}$$
$$|A| \times \max(B) = 11$$

and N can be written as $100a + b$ where a, b are positive integers less than 100, find $a + b$.

21. For $n \in \mathbb{N}$, consider non-negative integer-valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n} (i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

22. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

23. In the coordinate plane, a point is called a *lattice point* if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$.

24. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set $\{5, 6, 7, 8, 9, 10\}$.

25. Find the least positive integer n such that there are at least 1000 unordered pairs of diagonals in a regular polygon with n vertices that intersect at a right angle in the interior of the polygon.

SPACE FOR ROUGH WORK

26. In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations $1, 2, 2^2, 2^3, \ldots$ Bens. The rules of the Government of Binary stipulate that one can not use more than two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government.

27. A quadruple (a, b, c, d) of distinct integers is said to be *balanced* if $a + c = b + d$. Let $\mathcal{S}$ be any set of quadruples (a, b, c, d) where $1 \leq a < b < d < c \leq 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in $\mathcal{S}$.

28. On each side of an equilateral triangle with side length n units, where n is an integer, $1 \leq n \leq 100$, consider $n - 1$ points that divide the side into n equal segments. Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up. Two coins are said to be *adjacent* if the distance between them is 1 unit. A *move* consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.

SPACE FOR ROUGH WORK

29. A positive integer $n > 1$ is called *beautiful* if n can be written in one and only one way as $n = a_1 + a_2 + \cdots + a_k = a_1 \cdot a_2 \cdots a_k$ for some positive integers $a_1, a_2, \ldots, a_k$, where $k > 1$ and $a_1 \geq a_2 \geq \cdots \geq a_k$. (For example 6 is beautiful since $6 = 3 \cdot 2 \cdot 1 = 3 + 2 + 1$, and this is unique. But 8 is not beautiful since $8 = 4 + 2 + 1 + 1 = 4 \cdot 2 \cdot 1 \cdot 1$ as well as $8 = 2 + 2 + 2 + 1 + 1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$, so uniqueness is lost.) Find the largest beautiful number less than 100.

30. Let $d(m)$ denote the number of positive integer divisors of a positive integer m. If r is the number of integers $n \leq 2023$ for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r.

SPACE FOR ROUGH WORK

Answers

QNo	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Answer	22	54	23	07	10	16	48	59	17	51	14	18	58	40	03
QNo	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Answer	94	66	71	92	43	15	77	18	31	28	19	91	67	95	18

Clarifications for Students' Queries

Queries were received on questions 7, 9, 19, 21, 23 and 27. We have given below detailed clarifications to these queries. Some other queries on correctness of answers for a few other questions were received. We confirm that the answers given in our web-site are correct. The students who have such concerns can re-check their solutions themselves.

7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is coloured either red or yellow with opposite faces always of the same colour. The dice are distinguished by the colour pattern as well as the numbering pattern. Find the number of distinct dice that can be designed?

   The solutions submitted claiming that the answer must be 24 are incorrect.

   Since 1 and 2 are always opposite, the arrangement of 3,4,5,6 decides the number pattern. Each such arrangement corresponds to a circular permutation of 3,4,5,6. Hence there are $3! = 6$ such arrangements. Also, each pair of opposite faces have the same colour and there are 2 colours for use. Thus, the number of distinct designs is $6 \times 2^3 = 48$.

9. Find the number of triples (a, b, c) of positive integers such that

   (a) ab is a prime;

   (b) bc is a product of two primes;

   (c) abc is not divisible by square of any prime and

   (d) $abc \leq 30$.

   The query was that whether the triples are ordered or not is not specified. But we are counting triples (a, b, c) that satisfy the conditions and the notation clearly means that the triples are ordered.

   The possible triples are

$$\begin{array}{lllll} (1,2,3) & (1,2,5) & (1,2,7) & (1,2,11) & (1,2,13) \\ (1,3,2) & (1,3,5) & (1,3,7) & (1,5,2) & (1,5,3) \\ (2,1,15) & (3,1,10) & (5,1,6) & (1,7,2) & (1,7,3) \\ (1,11,2) & (1,13,2) & & & \end{array}$$

   We have a total of 17 triples.

19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n. Consider the set

$$A = \{n \in \mathbb{N} : P(n) \text{ is non-zero, square free and } S(n) \text{ is a proper divisor of } P(n)\}.$$

Find the maximum possible number of digits of the numbers in A.

The query was that the term square-free is not understood. But square-free is a standard term in Number Theory. A number is square-free if it is not divisible by the square of any prime number.

21. For $n \in \mathbb{N}$, consider non-negative integer valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n} (i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

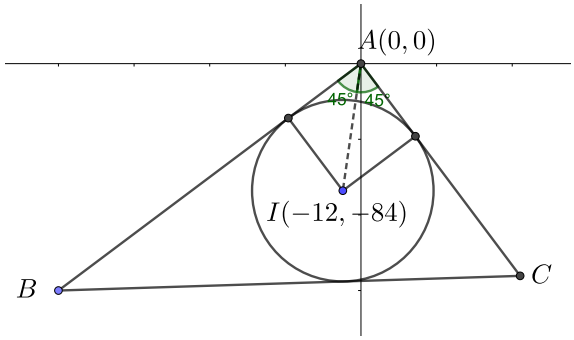
More than 90% of the queries were pertaining to this question claiming that there is ambiguity in the statement. It was argued that a function f *defined on* S meant that the domain and co-domain are both S. We have not seen such a definition of function anywhere. Also, surprisingly, almost all the mails on this question had the same text in the body of the mail.

However, there is **NO AMBIGUITY** in the statement. We are considering functions that are non-negative integer valued – hence functions that take values in non-negative integers (so that the co-domain is the set $\{0, 1, 2, \ldots\}$). The function is defined on $\{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Thus the domain of the function is the set $\{1, 2, \ldots, n\}$. When we say a function is defined **on** a set S, by standard convention, this means that the domain of the function is S. Note that it need not take values in the same set. When we say that a function **takes values in a set** T, then by standard convention, it means that the codomain of the function is the set T.

23. In the coordinate plane, a point is called a *lattice point* if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$.

It was claimed that the answer should be 24. But the solution presented by the students were erroneous. The correct solution is given below.

We will shift the vertex A to the origin. We need to find the number of triangles with vertices B, C at lattice points and incenter at $(-12, -84)$. The slope of AI



is $84/12 = 7$. If m is the slope of AC, then, since $\angle IAC = 45^\circ$, we have

$$\tan 45^\circ = \frac{m-7}{1+7m} \Rightarrow m = -\frac{4}{3}$$

Since $AB \perp AC$, slope of AB is $\frac{3}{4}$. Hence we can write B as $(-4t, -3t)$ and C as $(3t', -4t')$ for some positive t, t'. Since B and C are lattice points, it follows that t, t' are integers.

The inradius of the triangle ABC is $r = AI\cos 45^\circ = 60$. Also, we have $BC = AB + AC - 2r$. Thus

$$\begin{aligned} 5\sqrt{t^2+t'^2} &= 5t + 5t' - 120 \\ \Rightarrow tt' - 24(t+t') + 288 &= 0 \\ \Rightarrow (t-24)(t'-24) &= 288 = 2^5 \cdot 3^2 \end{aligned}$$

Thus $t - 24$ can be any divisor of 288 and there are 18 divisors for 288.

The only possible negative values for $t - 24$ and $t' - 24$ are $(-16, -18)$ and $(-18, -16)$. It is easy to see that the resulting triangle for these choices do not contain $(-12, -84)$ as in-center. Also, such a choice violates the condition $BC = AB + AC - 2r$. Thus we need to count only the positive factors of 288. There are 18 divisors for 288 and hence there are 18 such triangles.

27. A quadruple (a, b, c, d) of distinct integers is said to be *balanced* if $a + c = b + d$. Let $\mathcal{S}$ be any set of quadruples (a, b, c, d) where $1 \le a < b < d < c \le 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in $\mathcal{S}$.

It was claimed that the term cardinality was not defined. However, we maintain that cardinality of a set is a common term defined in set theory. Even the dictionary definition is "number of members in a set or a group".